本科１期６月度

解答

Ｚ会東大進学教室

高１難関大数学

問題

【1】(1)

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$= \boldsymbol{\frac{\sqrt{6}-\sqrt{2}}{4}} \quad (\text{答})$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \boldsymbol{\frac{\sqrt{6}+\sqrt{2}}{4}} \quad (\text{答})$$

$$\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{(\sqrt{6}-\sqrt{2})^2}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$$
$$= \boldsymbol{2-\sqrt{3}} \quad (\text{答})$$

＜別解＞

正接の加法定理を用いると

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}}$$
$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{2}$$
$$= \boldsymbol{2-\sqrt{3}} \quad (\text{答})$$

(2)

$$\sin 105^\circ = \sin(45^\circ + 60^\circ) = \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{2} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$
$$= \boldsymbol{\frac{\sqrt{2}+\sqrt{6}}{4}} \quad (\text{答})$$

$$\cos 105^\circ = \cos(45^\circ + 60^\circ) = \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$
$$= \boldsymbol{\frac{\sqrt{2}-\sqrt{6}}{4}} \quad (\text{答})$$

$$\tan 105^\circ = \frac{\sin 105^\circ}{\cos 105^\circ} = \frac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}} = \frac{(\sqrt{2}+\sqrt{6})^2}{(\sqrt{2}-\sqrt{6})(\sqrt{2}+\sqrt{6})}$$
$$= \boldsymbol{-2-\sqrt{3}} \quad (\text{答})$$

＜別解＞

正接の加法定理を用いると

$$\tan 105^\circ = \tan(45^\circ + 60^\circ) = \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ} = \frac{1+\sqrt{3}}{1 - 1 \times \sqrt{3}}$$
$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} = \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{4+2\sqrt{3}}{-2}$$
$$= \mathbf{-2-\sqrt{3}} \quad (答)$$

＜コメント＞

(1) の結果から，次のように導いてもよい．

$$\sin 105^\circ = \sin(90^\circ + 15^\circ) = \cos 15^\circ = \mathbf{\frac{\sqrt{6}+\sqrt{2}}{4}} \quad (答)$$

$$\cos 105^\circ = \cos(90^\circ + 15^\circ) = -\sin 15^\circ = \mathbf{\frac{-\sqrt{6}+\sqrt{2}}{4}} \quad (答)$$

$$\tan 105^\circ = \tan(90^\circ + 15^\circ) = -\frac{1}{\tan 15^\circ} = -\frac{1}{2-\sqrt{3}} = \mathbf{-2-\sqrt{3}} \quad (答)$$

(3)
$$\sin 195^\circ = \sin(135^\circ + 60^\circ) = \sin 135^\circ \cos 60^\circ + \cos 135^\circ \sin 60^\circ$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right) \times \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$
$$= \mathbf{\frac{\sqrt{2}-\sqrt{6}}{4}} \quad (答)$$

$$\cos 195^\circ = \cos(135^\circ + 60^\circ) = \cos 135^\circ \cos 60^\circ - \sin 135^\circ \sin 60^\circ$$
$$= -\frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{-1-\sqrt{3}}{2\sqrt{2}}$$
$$= -\mathbf{\frac{\sqrt{2}+\sqrt{6}}{4}} \quad (答)$$

$$\tan 195^\circ = \frac{\sin 195^\circ}{\cos 195^\circ} = \frac{\frac{\sqrt{2}-\sqrt{6}}{4}}{-\frac{\sqrt{2}+\sqrt{6}}{4}} = \frac{(\sqrt{2}-\sqrt{6})^2}{-(\sqrt{2}+\sqrt{6})(\sqrt{2}-\sqrt{6})}$$
$$= \mathbf{2-\sqrt{3}} \quad (答)$$

(4)
$$\sin \frac{11}{12}\pi = \sin\left(\frac{2}{3}\pi + \frac{\pi}{4}\right) = \sin \frac{2}{3}\pi \cos \frac{\pi}{4} + \cos \frac{2}{3}\pi \sin \frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \left(-\frac{1}{2}\right) \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$= \mathbf{\frac{\sqrt{6}-\sqrt{2}}{4}} \quad (答)$$

$$\cos \frac{11}{12}\pi = \cos\left(\frac{2}{3}\pi + \frac{\pi}{4}\right) = \cos \frac{2}{3}\pi \cos \frac{\pi}{4} - \sin \frac{2}{3}\pi \sin \frac{\pi}{4}$$
$$= \left(-\frac{1}{2}\right) \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{-1-\sqrt{3}}{2\sqrt{2}}$$
$$= -\mathbf{\frac{\sqrt{6}+\sqrt{2}}{4}} \quad (答)$$

$$\tan\frac{11}{12}\pi=\frac{\sin\frac{11}{12}\pi}{\cos\frac{11}{12}\pi}=\frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{-\frac{\sqrt{6}+\sqrt{2}}{4}}=\frac{(\sqrt{6}-\sqrt{2})^2}{-(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$$

$$=\boldsymbol{\sqrt{3}-2}\qquad(答)$$

(5) $\dfrac{\pi}{8}=\dfrac{\frac{\pi}{4}}{2}$

であることから，半角の公式よりそれぞれ

$$\sin^2\frac{\pi}{8}=\sin^2\frac{\frac{\pi}{4}}{2}=\frac{1-\cos\frac{\pi}{4}}{2}=\frac{1-\frac{1}{\sqrt{2}}}{2}=\frac{2-\sqrt{2}}{4}$$

$$\cos^2\frac{\pi}{8}=\cos^2\frac{\frac{\pi}{4}}{2}=\frac{1+\cos\frac{\pi}{4}}{2}=\frac{1+\frac{1}{\sqrt{2}}}{2}=\frac{2+\sqrt{2}}{4}$$

より，$\dfrac{\pi}{8}$ は第 1 象限の角であることを考慮すると

$$\sin\frac{\pi}{8}=\boldsymbol{\frac{\sqrt{2-\sqrt{2}}}{2}}\qquad(答)$$

$$\cos\frac{\pi}{8}=\boldsymbol{\frac{\sqrt{2+\sqrt{2}}}{2}}\qquad(答)$$

$$\therefore\ \tan\frac{\pi}{8}=\frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}}=\frac{\frac{\sqrt{2-\sqrt{2}}}{2}}{\frac{\sqrt{2+\sqrt{2}}}{2}}=\sqrt{\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}}=\sqrt{3-2\sqrt{2}}$$

$$=\boldsymbol{\sqrt{2}-1}\qquad(答)$$

(6) $67.5^\circ=\dfrac{135^\circ}{2}$

であるから，半角の公式よりそれぞれ

$$\sin^2 67.5^\circ=\sin^2\frac{135^\circ}{2}=\frac{1-\cos 135^\circ}{2}=\frac{1-\left(-\frac{1}{\sqrt{2}}\right)}{2}=\frac{\sqrt{2}+1}{2\sqrt{2}}$$

$$=\frac{2+\sqrt{2}}{4}$$

$$\cos^2 67.5^\circ=\cos^2\frac{135^\circ}{2}=\frac{1+\cos 135^\circ}{2}=\frac{1+\left(-\frac{1}{\sqrt{2}}\right)}{2}=\frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$=\frac{2-\sqrt{2}}{4}$$

より，67.5° が第 1 象限の角であることを考慮すると

$$\sin 67.5^\circ = \boldsymbol{\frac{\sqrt{2+\sqrt{2}}}{2}} \qquad \text{(答)}$$

$$\cos 67.5^\circ = \boldsymbol{\frac{\sqrt{2-\sqrt{2}}}{2}} \qquad \text{(答)}$$

$$\therefore\ \tan 67.5^\circ = \frac{\sin 67.5^\circ}{\cos 67.5^\circ} = \frac{\dfrac{\sqrt{2+\sqrt{2}}}{2}}{\dfrac{\sqrt{2-\sqrt{2}}}{2}} = \sqrt{\frac{(2+\sqrt{2})^2}{(2-\sqrt{2})(2+\sqrt{2})}}$$

$$= \sqrt{3+2\sqrt{2}} = \boldsymbol{\sqrt{2}+1} \qquad \text{(答)}$$

(7) $\dfrac{41}{12}\pi = 4\pi - \dfrac{7}{12}\pi = 4\pi - \left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$

より

$$\begin{aligned}\sin \frac{41}{12}\pi &= \sin\left\{4\pi - \left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right\} \\ &= -\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = -\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= -\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{2}\times\frac{1}{\sqrt{2}} = -\frac{\sqrt{3}+1}{2\sqrt{2}} \\ &= \boldsymbol{-\frac{\sqrt{2}+\sqrt{6}}{4}} \qquad \text{(答)}\end{aligned}$$

$$\begin{aligned}\cos \frac{41}{12}\pi &= \cos\left\{4\pi - \left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right\} \\ &= \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} \\ &= \boldsymbol{\frac{\sqrt{2}-\sqrt{6}}{4}} \qquad \text{(答)}\end{aligned}$$

$$\begin{aligned}\tan \frac{41}{12}\pi &= \frac{\sin\dfrac{41}{12}\pi}{\cos\dfrac{41}{12}\pi} = \frac{-\dfrac{\sqrt{2}+\sqrt{6}}{4}}{\dfrac{\sqrt{2}-\sqrt{6}}{4}} = \frac{-(\sqrt{2}+\sqrt{6})^2}{(\sqrt{2}-\sqrt{6})(\sqrt{2}+\sqrt{6})} \\ &= \boldsymbol{2+\sqrt{3}} \qquad \text{(答)}\end{aligned}$$

(8) $450^\circ = 360^\circ + 90^\circ$

であるから

$\sin 450^\circ = \sin 90^\circ = \boldsymbol{1}$ (答)
$\cos 450^\circ = \cos 90^\circ = \boldsymbol{0}$ (答)
$\tan 450^\circ = \tan 90^\circ$ より $\tan 450^\circ$ の値は**存在しない** (答)

【2】(1)　$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ において，α を $90^\circ-\alpha$ とおきかえると

$\cos\{(90^\circ-\alpha)-\beta\}=\cos(90^\circ-\alpha)\cos\beta+\sin(90^\circ-\alpha)\sin\beta$

ここで，$\sin(90^\circ-\alpha)=\cos\alpha$，$\cos(90^\circ-\alpha)=\sin\alpha$ だから

(左辺) $=\cos\{(90^\circ-(\alpha+\beta)\}=\sin(\alpha+\beta)$
(右辺) $=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

ゆえに

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$　〔証明終〕

(2)　$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ において，α を $90^\circ-\alpha$ とおきかえると

$\cos\{(90^\circ-\alpha)+\beta\}=\cos(90^\circ-\alpha)\cos\beta-\sin(90^\circ-\alpha)\sin\beta$

よって，(1) と同様にして

(左辺) $=\cos\{(90^\circ-(\alpha-\beta)\}=\sin(\alpha-\beta)$
(右辺) $=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

ゆえに

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$　〔証明終〕

＜コメント＞
(2) は，(1) において，β を $-\beta$ とおきかえて導くこともできる.

【3】

$$\begin{aligned}&\sin 5^\circ\sin 125^\circ+\sin 5^\circ\sin 245^\circ+\sin 125^\circ\sin 245^\circ\\=&\sin 5^\circ\sin(120^\circ+5^\circ)+\sin 5^\circ\sin(240^\circ+5^\circ)+\sin(120^\circ+5^\circ)\sin(240^\circ+5^\circ)\end{aligned}$$

ここで，$a=\sin 5^\circ$，$b=\cos 5^\circ$ とおくと

$$\begin{aligned}(\text{与式})=&a(\sin 120^\circ\cdot b+\cos 120^\circ\cdot a)+a(\sin 240^\circ\cdot b+\cos 240^\circ\cdot a)\\&+(\sin 120^\circ\cdot b+\cos 120^\circ\cdot a)(\sin 240^\circ\cdot b+\cos 240^\circ\cdot a)\\=&a\left(\frac{\sqrt{3}}{2}b-\frac{1}{2}a\right)+a\left(-\frac{\sqrt{3}}{2}b-\frac{1}{2}a\right)+\left(\frac{\sqrt{3}}{2}b-\frac{1}{2}a\right)\left(-\frac{\sqrt{3}}{2}b-\frac{1}{2}a\right)\\=&-\frac{1}{2}a^2-\frac{1}{2}a^2-\frac{1}{4}(\sqrt{3}b-a)(\sqrt{3}b+a)\\=&-a^2-\frac{1}{4}(3b^2-a^2)\\=&-\frac{3}{4}(a^2+b^2)\\=&-\boldsymbol{\frac{3}{4}}\quad(\text{答})\end{aligned}$$

＜別解 1 ＞

$$
\begin{aligned}
&\sin 5^\circ \sin 125^\circ + \sin 5^\circ \sin 245^\circ + \sin 125^\circ \sin 245^\circ \\
&= -\frac{1}{2}\{\cos(5^\circ + 125^\circ) - \cos(5^\circ - 125^\circ)\} - \frac{1}{2}\{\cos(5^\circ + 245^\circ) - \cos(5^\circ - 245^\circ)\} \\
&\qquad - \frac{1}{2}\{\cos(125^\circ + 245^\circ) - \cos(125^\circ - 245^\circ)\} \\
&= -\frac{1}{2}\{\cos(120^\circ + 10^\circ) - \cos(-120^\circ)\} - \frac{1}{2}\{\cos 250^\circ - \cos(-240^\circ)\} \\
&\qquad - \frac{1}{2}\{\cos(360^\circ + 10^\circ) - \cos(-120^\circ)\} \\
&= -\frac{1}{2}\{\cos(120^\circ + 10^\circ) - \cos 120^\circ\} - \frac{1}{2}\{\cos(-120^\circ + 10^\circ) - \cos 120^\circ\} \\
&\qquad - \frac{1}{2}\{\cos 10^\circ - \cos 120^\circ\} \\
&= -\frac{1}{2}\{\cos(120^\circ + 10^\circ) + \cos(-120^\circ + 10^\circ) + \cos 10^\circ - 3\cos 120^\circ\} \\
&= -\frac{1}{2}\{(\cos 120^\circ \cos 10^\circ - \sin 120^\circ \sin 10^\circ) \\
&\qquad + (\cos(-120^\circ)\cos 10^\circ - \sin(-120^\circ)\sin 10^\circ) + \cos 10^\circ - 3\cos 120^\circ\} \\
&= -\frac{1}{2}(2\cos 120^\circ \cos 10^\circ + \cos 10^\circ - 3\cos 120^\circ) \\
&= -\frac{1}{2}\left\{2 \cdot \left(-\frac{1}{2}\right) \cdot \cos 10^\circ + \cos 10^\circ - 3 \cdot \left(-\frac{1}{2}\right)\right\} \\
&= -\boldsymbol{\frac{3}{4}} \qquad \text{(答)}
\end{aligned}
$$

＜別解 2 ＞

$$
\begin{aligned}
&\sin 5^\circ \sin 125^\circ + \sin 5^\circ \sin 245^\circ + \sin 125^\circ \sin 245^\circ \\
&= \sin 125^\circ \times (\sin 5^\circ + \sin 245^\circ) + \sin 5^\circ \sin 245^\circ \\
&= \sin 125^\circ \times \{\sin(125^\circ - 120^\circ) + \sin(125^\circ + 120^\circ)\} \\
&\qquad\qquad + \sin(125^\circ - 120^\circ)\sin(125^\circ + 120^\circ) \\
&= \sin 125^\circ \times \left\{-\frac{1}{2}(\sqrt{3}\cos 125^\circ + \sin 125^\circ) + \frac{1}{2}(\sqrt{3}\cos 125^\circ - \sin 125^\circ)\right\} \\
&\qquad + \left\{-\frac{1}{2}(\sqrt{3}\cos 125^\circ + \sin 125^\circ)\right\} \times \left\{\frac{1}{2}(\sqrt{3}\cos 125^\circ - \sin 125^\circ)\right\}
\end{aligned}
$$

ここで，$\cos 125^\circ = a$，$\sin 125^\circ = b$ とすると

$$
\begin{aligned}
\text{(与式)} &= b \times \left\{-\frac{1}{2}(\sqrt{3}a + b) + \frac{1}{2}(\sqrt{3}a - b)\right\} - \frac{1}{2}(\sqrt{3}a + b) \times \frac{1}{2}(\sqrt{3}a - b) \\
&= -\frac{1}{2}b(\sqrt{3}a + b - \sqrt{3}a + b) - \frac{1}{4}(3a^2 - b^2) \\
&= -\frac{3}{4}(a^2 + b^2) \\
&= -\boldsymbol{\frac{3}{4}} \qquad \text{(答)}
\end{aligned}
$$

【4】 $\cos^2\theta - \sin^2\left(\theta + \dfrac{\pi}{3}\right) + \cos^2\left(\theta + \dfrac{2}{3}\pi\right)$

$$
\begin{aligned}
&= \cos^2\theta - \left(\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right)^2 + \left(\cos\theta\cos\frac{2}{3}\pi - \sin\theta\sin\frac{2}{3}\pi\right)^2 \\
&= \cos^2\theta - \left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)^2 + \left(-\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)^2 \\
&= \cos^2\theta - \frac{1}{4}\left(\sin^2\theta + 3\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta\right) \\
&\qquad + \frac{1}{4}\left(\cos^2\theta + 3\sin^2\theta + 2\sqrt{3}\sin\theta\cos\theta\right) \\
&= \cos^2\theta + \frac{1}{2}\left(\sin^2\theta - \cos^2\theta\right) \\
&= \frac{1}{2}(\cos^2\theta + \sin^2\theta) \\
&= \boldsymbol{\frac{1}{2}} \qquad \text{(答)}
\end{aligned}
$$

【5】 $\left(\tan x - \dfrac{1}{\tan x}\right)^2 = \left(\tan x + \dfrac{1}{\tan x}\right)^2 - 4 = \left(\dfrac{17}{4}\right)^2 - 4 = \dfrac{225}{16}$

$0 < x \leqq \dfrac{\pi}{4}$ より， $0 < \tan x \leqq 1$ であるから

$$\tan x - \frac{1}{\tan x} = -\frac{15}{4}$$

これより

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = 2 \times \frac{1}{\frac{1}{\tan x} - \tan x} = 2 \times \frac{1}{\frac{15}{4}} = \boldsymbol{\frac{8}{15}} \qquad \text{(答)}$$

さらに

$$\cos^2 2x = \frac{1}{\tan^2 2x + 1} = \frac{1}{\frac{64}{225} + 1} = \frac{225}{289}$$

$0 < x \leqq \dfrac{\pi}{4}$ より， $0 < 2x \leqq \dfrac{\pi}{2}$ であるから

$$\cos 2x = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

$$\therefore\ \sin 2x = \tan 2x \times \cos 2x = \frac{8}{15} \times \frac{15}{17} = \frac{8}{17}$$

であることから

$$(\sin x + \cos x)^2 = 1 + 2\sin x\cos x = 1 + \sin 2x = 1 + \frac{8}{17} = \frac{25}{17}$$

$0 < x \leqq \dfrac{\pi}{4}$ より， $\sin x > 0$，$\cos x > 0$ であるから

$$\sin x + \cos x = \sqrt{\frac{25}{17}} = \boldsymbol{\frac{5}{17}\sqrt{17}} \qquad \text{(答)}$$

【6】(1) $\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$

$0 < \alpha < \frac{\pi}{2}$ より，$\cos\alpha > 0$ だから，$\cos\alpha = \frac{\sqrt{15}}{4}$

また

$\sin^2\beta = 1 - \cos^2\beta = 1 - \left(-\frac{1}{3}\right)^2 = \frac{8}{9}$

$\frac{\pi}{2} < \alpha < \pi$ より，$\sin\beta > 0$ だから，$\sin\beta = \frac{2\sqrt{2}}{3}$

したがって

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{1}{4}\times\left(-\frac{1}{3}\right) + \frac{\sqrt{15}}{4}\times\frac{2\sqrt{2}}{3} = -\frac{1}{12} + \frac{\sqrt{30}}{6} \\ &= \boldsymbol{\frac{-1+2\sqrt{30}}{12}} \quad \text{(答)}\end{aligned}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{\sqrt{15}}{4}\times\left(-\frac{1}{3}\right) - \frac{1}{4}\times\frac{2\sqrt{2}}{3} = -\frac{\sqrt{15}}{12} - \frac{\sqrt{2}}{6} \\ &= \boldsymbol{\frac{-\sqrt{15}-2\sqrt{2}}{12}} \quad \text{(答)}\end{aligned}$$

(2) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{2+(-2)}{1-2\times(-2)} = \mathbf{0}$ (答)

$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$ より，$\cos^2\alpha = \frac{1}{2^2+1} = \frac{1}{5}$

$0 < \alpha < \frac{\pi}{2}$ より，$\cos\alpha > 0$ だから，$\cos\alpha = \frac{1}{\sqrt{5}}$

よって

$\sin\alpha = \tan\alpha\cos\alpha = 2\times\frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}}$

$\tan^2\beta + 1 = \frac{1}{\cos^2\beta}$ より，$\cos^2\beta = \frac{1}{(-2)^2+1} = \frac{1}{5}$

$\frac{\pi}{2} < \beta < \pi$ より，$\cos\beta < 0$ だから，$\cos\beta = -\frac{1}{\sqrt{5}}$

よって

$\sin\beta = \tan\beta\cos\beta = -2\times\left(-\frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}}$

したがって

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= \frac{2}{\sqrt{5}}\times\left(-\frac{1}{\sqrt{5}}\right) - \frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}} = -\frac{2}{5} - \frac{2}{5} \\ &= \boldsymbol{-\frac{4}{5}} \quad \text{(答)}\end{aligned}$$

(3)　$\sin\alpha-\sin\beta=\dfrac{1}{2}$ の両辺を 2 乗して

$\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta=\dfrac{1}{4}\quad\cdots\cdots①$

$\cos\alpha+\cos\beta=\dfrac{1}{3}$ の両辺を 2 乗して

$\cos^2\alpha+2\cos\alpha\cos\beta+\cos^2\beta=\dfrac{1}{9}\quad\cdots\cdots②$

①＋② より

$(\sin^2\alpha+\cos^2\alpha)+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)+(\sin^2\beta+\cos^2\beta)=\dfrac{13}{36}$

$1+2\cos(\alpha+\beta)+1=\dfrac{13}{36}$

よって，$\cos(\alpha+\beta)=-\mathbf{\dfrac{59}{72}}$　(答)

【7】(1)　$\cos^2\alpha=1-\sin^2\alpha=1-\left(\dfrac{3}{5}\right)^2=\dfrac{16}{25}$

ここで，$0<\alpha<\dfrac{\pi}{2}$ より，$\cos\alpha>0$ だから，$\cos\alpha=\dfrac{4}{5}$ であり，また

$\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{\frac{3}{5}}{\frac{4}{5}}=\dfrac{3}{4}$ である．

したがって

$\sin2\alpha=2\sin\alpha\cos\alpha=2\times\dfrac{3}{5}\times\dfrac{4}{5}=\mathbf{\dfrac{24}{25}}$　(答)

$\cos2\alpha=\cos^2\alpha-\sin^2\alpha=\left(\dfrac{4}{5}\right)^2-\left(\dfrac{3}{5}\right)^2=\mathbf{\dfrac{7}{25}}$　(答)

$\tan2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}=\mathbf{\dfrac{24}{7}}$　(答)

$0<\alpha<\dfrac{\pi}{2}$ より，$0<\dfrac{\alpha}{2}<\dfrac{\pi}{4}$ だから，$\sin\dfrac{\alpha}{2}>0$，$\cos\dfrac{\alpha}{2}>0$，$\tan\dfrac{\alpha}{2}>0$ であることに注意して

$\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}=\dfrac{1-\frac{4}{5}}{2}=\dfrac{1}{10}$

よって，$\sin\dfrac{\alpha}{2}=\mathbf{\dfrac{1}{\sqrt{10}}}$　(答)

$\cos^2\dfrac{\alpha}{2}=\dfrac{1+\cos\alpha}{2}=\dfrac{1+\frac{4}{5}}{2}=\dfrac{9}{10}$

よって，$\cos\dfrac{\alpha}{2}=\mathbf{\dfrac{3}{\sqrt{10}}}$　(答)

$\tan^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{1+\cos\alpha}=\dfrac{1-\frac{4}{5}}{1+\frac{4}{5}}=\dfrac{1}{9}$

よって，$\tan\dfrac{\alpha}{2}=\mathbf{\dfrac{1}{3}}$　　(答)

(2)　$\tan^2\beta+1=\dfrac{1}{\cos^2\beta}$ より，$\left(-\dfrac{5}{12}\right)^2+1=\dfrac{1}{\cos^2\beta}$

よって，$\cos^2\beta=\dfrac{144}{169}$

ここで，$\dfrac{\pi}{2}<\beta<\pi$ より，$\cos\beta<0$ だから，$\cos\beta=-\dfrac{12}{13}$

また，$\sin\beta=\tan\beta\cos\beta=-\dfrac{5}{12}\times\left(-\dfrac{12}{13}\right)=\dfrac{5}{13}$

したがって

$\sin 2\beta=2\sin\beta\cos\beta=2\times\dfrac{5}{13}\times\left(-\dfrac{12}{13}\right)=-\mathbf{\dfrac{120}{169}}$　　(答)

$\cos 2\beta=\cos^2\beta-\sin^2\beta=\left(-\dfrac{12}{13}\right)^2-\left(\dfrac{5}{13}\right)^2=\mathbf{\dfrac{119}{169}}$　　(答)

$$\tan 2\beta=\frac{2\tan\beta}{1-\tan^2\beta}=\frac{2\times\left(-\frac{5}{12}\right)}{1-\left(-\frac{5}{12}\right)^2}=-\mathbf{\frac{120}{119}}$$　　(答)

$\dfrac{\pi}{2}<\beta<\pi$ より，$\dfrac{\pi}{4}<\dfrac{\beta}{2}<\dfrac{\pi}{2}$ だから，

$\sin\dfrac{\beta}{2}>0$，$\cos\dfrac{\beta}{2}>0$，$\tan\dfrac{\beta}{2}>0$ であることに注意して

$$\sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2}=\frac{1-\left(-\frac{12}{13}\right)}{2}=\frac{25}{26}$$

よって，$\sin\dfrac{\beta}{2}=\mathbf{\dfrac{5}{\sqrt{26}}}$　　(答)

$$\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2}=\frac{1+\left(-\frac{12}{13}\right)}{2}=\frac{1}{26}$$

よって，$\cos\dfrac{\beta}{2}=\mathbf{\dfrac{1}{\sqrt{26}}}$　　(答)

$$\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta}=\frac{1-\left(-\frac{12}{13}\right)}{1+\left(-\frac{12}{13}\right)}=25$$

よって，$\tan\dfrac{\beta}{2}=\mathbf{5}$　　(答)

【8】(1)　2 直線のなす角を θ $\left(0\leqq\theta\leqq\dfrac{\pi}{2}\right)$ とすると，$3\sqrt{3}\times\dfrac{2}{\sqrt{3}}=6\fallingdotseq-1$ より，$\theta\fallingdotseq\dfrac{\pi}{2}$ だから

$$\tan\theta=\left|\frac{3\sqrt{3}-\frac{2}{\sqrt{3}}}{1+3\sqrt{3}\times\frac{2}{\sqrt{3}}}\right|=\left|\frac{3\sqrt{3}-\frac{2}{3}\sqrt{3}}{1+6}\right|=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$$

よって，$\theta=\mathbf{\dfrac{\pi}{6}}$　　(答)

(2) $3x+y+1=0$ より，$y=-3x-1$

$x+2y-1=0$ より，$y=-\frac{1}{2}x+\frac{1}{2}$

2 直線のなす角を θ $\left(0 \leqq \theta \leqq \frac{\pi}{2}\right)$ とすると，$-3\times\left(-\frac{1}{2}\right)=\frac{3}{2}\neq -1$ より，$\theta \neq \frac{\pi}{2}$ だから

$$\tan\theta=\left|\frac{-3-\left(-\frac{1}{2}\right)}{1+(-3)\times\left(-\frac{1}{2}\right)}\right|=\left|\frac{-\frac{5}{2}}{\frac{5}{2}}\right|=1$$

よって，$\theta=\boldsymbol{\frac{\pi}{4}}$ **(答)**

【9】(1) $X=\sin x$ とおくと，(ただし，$-1\leqq X\leqq 1$)

$\cos 2x = 1-2\sin^2 x = 1-2X^2$

であることから，与えられた方程式は

$$\begin{aligned}\cos 2x-\sin x&=0\\1-2X^2-X&=0\\2X^2+X-1&=0\\(2X-1)(X+1)&=0\\\therefore X&=-1,\ \frac{1}{2}\end{aligned}$$

よって，$\sin x=-1,\ \frac{1}{2}$ より，

$x=\boldsymbol{\frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{3}{2}\pi}$ **(答)**

(2) $X=\tan x$ とおくと，

$\tan 2x=\frac{2\tan x}{1-\tan^2 x}=\frac{2X}{1-X^2}$

となる．よって，

$$\begin{aligned}\tan 2x+\tan x&=0\\\frac{2X}{1-X^2}+X&=0\\2X+X(1-X^2)&=0\\X^3-3X&=0\\X\left(X-\sqrt{3}\right)\left(X+\sqrt{3}\right)&=0\end{aligned}$$

よって，$\tan x=0,\ \pm\sqrt{3}$ より，

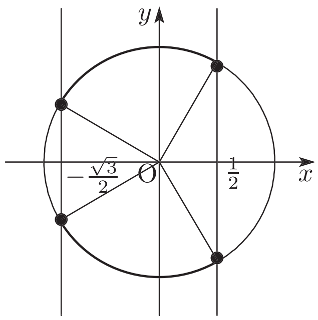
$x=\boldsymbol{0,\ \frac{\pi}{3},\ \frac{2}{3}\pi,\ \pi,\ \frac{4}{3}\pi,\ \frac{5}{3}\pi}$ **(答)**

(3) $X=\cos x$ とおくと，(ただし，$-1\leqq X\leqq 1$)

$\cos 2x=2\cos^2 x-1=2X^2-1$

であるから，

$$2\cos 2x + 2\left(\sqrt{3}-1\right)\cos x + 2 - \sqrt{3} \leqq 0$$
$$2(2X^2-1) + 2\left(\sqrt{3}-1\right)X + 2 - \sqrt{3} \leqq 0$$
$$4X^2 + 2\left(\sqrt{3}-1\right)X - \sqrt{3} \leqq 0$$
$$(2X-1)\left(2X+\sqrt{3}\right) \leqq 0$$
$$-\frac{\sqrt{3}}{2} \leqq X \leqq \frac{1}{2}$$



よって，$-\frac{\sqrt{3}}{2} \leqq \cos x \leqq \frac{1}{2}$ をみたす x の値の範囲は，右図より，

$$\boldsymbol{\frac{\pi}{3} \leqq x \leqq \frac{5}{6}\pi,\ \frac{7}{6}\pi \leqq x \leqq \frac{5}{3}\pi}$$ **(答)**

(4) $\sin 2x = 2\sin x\cos x$ であるので，

$$\sin 2x - \sqrt{3}\sin x > 0$$
$$2\sin x\cos x - \sqrt{3}\sin x > 0$$
$$\sin x\left(2\cos x - \sqrt{3}\right) > 0$$

よって，求める条件は，

「$\sin x > 0$　かつ　$\cos x > \frac{\sqrt{3}}{2}$」または

「$\sin x < 0$　かつ　$\cos x < \frac{\sqrt{3}}{2}$」

である．

(i) $\sin x > 0$　かつ　$\cos x > \frac{\sqrt{3}}{2}$ のとき，

下図 1 より，$0 < x < \frac{\pi}{6}$

(ii) $\sin x < 0$　かつ　$\cos x < \frac{\sqrt{3}}{2}$ のとき，

下図 2 より，$\pi < x < \frac{11}{6}\pi$

(i)，(ii) より，求める x の値の範囲は，

$$\boldsymbol{0 < x < \frac{\pi}{6},\ \pi < x < \frac{11}{6}\pi}$$ **(答)**

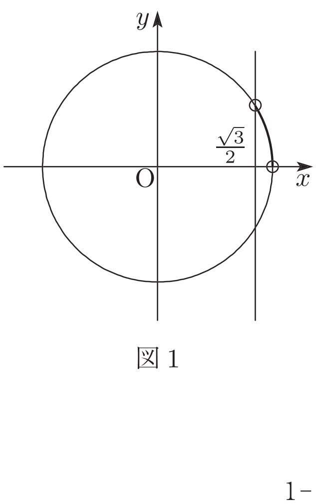


図 1

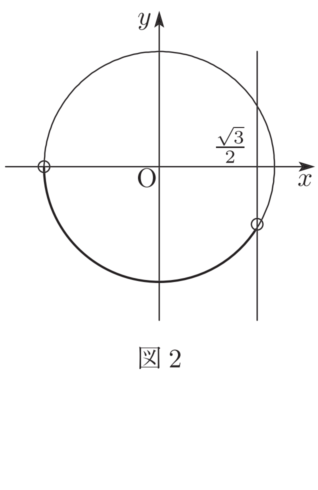


図 2

添削課題

【1】(1)　(i) $0<\theta<\dfrac{\pi}{2}$ より，$\sin\theta>0$ であるので

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\frac{2}{3}\right)^2}=\boldsymbol{\frac{\sqrt{5}}{3}}\qquad\text{(答)}$$

(ii)

$$\begin{aligned}\sin\left(\theta+\frac{\pi}{3}\right)&=\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}\\&=\frac{\sqrt{5}}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{\sqrt{3}}{2}\\&=\boldsymbol{\frac{\sqrt{5}+2\sqrt{3}}{6}}\qquad\text{(答)}\end{aligned}$$

(2) $\dfrac{5}{12}\pi=\dfrac{\pi}{4}+\dfrac{\pi}{6}$ を考え，

$$\begin{aligned}\cos\frac{5}{12}\pi&=\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\&=\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}\\&=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\&=\boldsymbol{\frac{\sqrt{6}-\sqrt{2}}{4}}\qquad\text{(答)}\end{aligned}$$

【2】(1) $0<\alpha<\dfrac{\pi}{2}$，$0<\beta<\dfrac{\pi}{2}$ より，$\cos\alpha>0$，$\cos\beta>0$ であるから，

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{2}{3}\right)^2}=\frac{\sqrt{5}}{3}$$
$$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\left(\frac{1}{3}\right)^2}=\frac{2\sqrt{2}}{3}$$

以上より，

$$\begin{aligned}\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\&=\frac{2}{3}\cdot\frac{2\sqrt{2}}{3}+\frac{\sqrt{5}}{3}\cdot\frac{1}{3}\\&=\boldsymbol{\frac{4\sqrt{2}+\sqrt{5}}{9}}\qquad\text{(答)}\end{aligned}$$

$$\begin{aligned}\cos(\alpha+\beta)&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\&=\frac{\sqrt{5}}{3}\cdot\frac{2\sqrt{2}}{3}-\frac{2}{3}\cdot\frac{1}{3}\\&=\boldsymbol{\frac{2\sqrt{10}-2}{9}}\qquad\text{(答)}\end{aligned}$$

(2) $0 < \alpha < \dfrac{\pi}{2}$, $0 < \beta < \dfrac{\pi}{2}$ より，$\tan\alpha > 0$, $\tan\beta > 0$ であるから，

$$\tan\alpha = \sqrt{\frac{1}{\cos^2\alpha} - 1} = \sqrt{\frac{1}{\left(\frac{2}{3}\right)^2} - 1} = \sqrt{\frac{9}{4} - 1} = \frac{\sqrt{5}}{2}$$
$$\tan\beta = \sqrt{\frac{1}{\cos^2\beta} - 1} = \sqrt{\frac{1}{\left(\frac{\sqrt{3}}{3}\right)^2} - 1} = \sqrt{3-1} = \sqrt{2}$$

以上より，

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{\sqrt{5}}{2} + \sqrt{2}}{1 - \frac{\sqrt{5}}{2}\cdot\sqrt{2}} = -\boldsymbol{\frac{3\sqrt{2}+2\sqrt{5}}{2}} \qquad \text{(答)}$$

【3】(1) $0 < \theta < \dfrac{\pi}{2}$ より，$\cos\theta > 0$ であるから，

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

したがって

$$\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{4}{5} \times \frac{3}{5} = \boldsymbol{\frac{24}{25}} \qquad \text{(答)}$$
$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2 \times \left(\frac{4}{5}\right)^2 = -\boldsymbol{\frac{7}{25}} \qquad \text{(答)}$$

また，$0 < \theta < \dfrac{\pi}{2}$ より，$0 < \dfrac{\theta}{2} < \dfrac{\pi}{4}$ で，$\sin\dfrac{\theta}{2} > 0$, $\cos\dfrac{\theta}{2} > 0$ であるから，

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} = \frac{1-\frac{3}{5}}{2} = \frac{1}{5} \qquad \therefore\ \sin\frac{\theta}{2} = \boldsymbol{\frac{1}{\sqrt{5}}} \qquad \text{(答)}$$
$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} = \frac{1+\frac{3}{5}}{2} = \frac{4}{5} \qquad \therefore\ \cos\frac{\theta}{2} = \boldsymbol{\frac{2}{\sqrt{5}}} \qquad \text{(答)}$$

(2) $t^2 = \tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$ より，

$$\begin{aligned}(1+\cos\theta)t^2 = 1 - \cos\theta \quad &\Longleftrightarrow \quad t^2 + t^2\cos\theta = 1 - \cos\theta \\ &\Longleftrightarrow \quad (t^2+1)\cos\theta = 1 - t^2 \\ &\therefore\ \boldsymbol{\cos\theta = \frac{1-t^2}{1+t^2}} \qquad \text{(答)}\end{aligned}$$

また，倍角の公式より

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \boldsymbol{\frac{2t}{1-t^2}} \qquad \text{(答)}$$

であるから，

$$\sin\theta = \cos\theta\tan\theta = \frac{1-t^2}{1+t^2}\cdot\frac{2t}{1-t^2} = \boldsymbol{\frac{2t}{1+t^2}} \qquad \text{(答)}$$

【4】(1) sin についての加法定理より，

$$\sin(\alpha+\beta) = \boxed{\boldsymbol{\sin\alpha\cos\beta}} + \boxed{\boldsymbol{\cos\alpha\sin\beta}} \quad \cdots ①$$
$$\sin(\alpha-\beta) = \boxed{\boldsymbol{\sin\alpha\cos\beta}} - \boxed{\boldsymbol{\cos\alpha\sin\beta}} \quad \cdots ②$$

① + ② より，

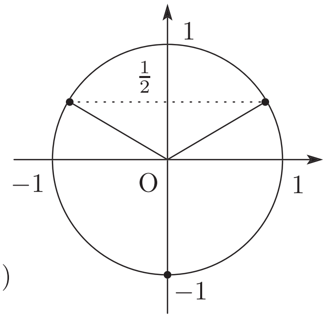
$$\sin(\alpha+\beta)+\sin(\alpha-\beta) = \boxed{\boldsymbol{2\sin\alpha\cos\beta}}$$

両辺を 2 で割って，左辺と右辺を入れ替えて

$$\therefore\ \sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$$

(2) (i) $\cos 2\theta = 1 - 2\sin^2\theta$ より，

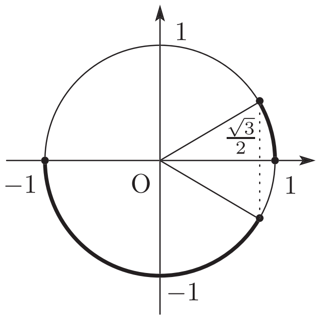
$$\begin{aligned} & \sin\theta = \cos 2\theta \\ \iff\ & \sin\theta = 1 - 2\sin^2\theta \\ \iff\ & 2\sin^2\theta + \sin\theta - 1 = 0 \\ \iff\ & (2\sin\theta - 1)(\sin\theta + 1) = 0 \\ \iff\ & \sin\theta = \frac{1}{2},\ -1 \\ & \therefore\ \boldsymbol{\theta = \frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{3}{2}\pi} \end{aligned}$$
(答)



(ii) $\sin 2\theta = 2\sin\theta\cos\theta$ より，

$$\begin{aligned} & \sin 2\theta - \sqrt{3}\sin\theta \geqq 0 \\ \iff\ & 2\sin\theta\cos\theta - \sqrt{3}\sin\theta \geqq 0 \\ \iff\ & \sin\theta(2\cos\theta - \sqrt{3}) \geqq 0 \end{aligned}$$

$$\therefore \begin{cases} \sin\theta \geqq 0 \quad \text{かつ} \quad \cos\theta \geqq \dfrac{\sqrt{3}}{2} \\ \text{または} \\ \sin\theta \leqq 0 \quad \text{かつ} \quad \cos\theta \leqq \dfrac{\sqrt{3}}{2} \end{cases}$$



したがって，

$$\boldsymbol{0 \leqq \theta \leqq \frac{\pi}{6},\ \pi \leqq \theta \leqq \frac{11}{6}\pi}$$
(答)

問題

【1】(1) P$(\sqrt{3}, 1)$ とすると

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2,\ \alpha = \frac{\pi}{6}$$

だから

$$\sqrt{3}\sin\theta + \cos\theta = \mathbf{2\sin\left(\boldsymbol{\theta} + \frac{\boldsymbol{\pi}}{6}\right)} \qquad (\text{答})$$

(2) P$(-1, 1)$ とすると

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2},\ \alpha = \frac{3}{4}\pi$$

だから

$$-\sin\theta + \cos\theta = \mathbf{\sqrt{2}\sin\left(\boldsymbol{\theta} + \frac{3}{4}\boldsymbol{\pi}\right)} \qquad (\text{答})$$

(3) P$(2, -3)$ とすると，$r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$ だから

$$2\sin\theta - 3\cos\theta = \mathbf{\sqrt{13}\sin(\boldsymbol{\theta} + \boldsymbol{\alpha})}$$

$$\left(\text{ただし，}\mathbf{\sin\boldsymbol{\alpha} = -\frac{3}{\sqrt{13}}},\ \mathbf{\cos\boldsymbol{\alpha} = \frac{2}{\sqrt{13}}}\right) \qquad (\text{答})$$

(4)
$$\begin{aligned}\sin(\theta + 30^\circ) - \cos\theta &= \sin\theta\cos 30^\circ + \cos\theta\sin 30^\circ - \cos\theta \\ &= \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \cos\theta \\ &= \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\end{aligned}$$

$\mathrm{P}\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ とすると

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1,\ \alpha = \frac{11}{6}\pi$$

だから

$$\begin{aligned}\sin\left(\theta + \frac{\pi}{6}\right) - \cos\theta &= \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta \\ &= \mathbf{\sin\left(\boldsymbol{\theta} + \frac{11}{6}\boldsymbol{\pi}\right)} \qquad (\text{答})\end{aligned}$$

(5)
$$\begin{aligned}\sqrt{2}\sin\theta + 2\sin\left(\theta + \frac{\pi}{4}\right) &= \sqrt{2}\sin\theta + 2\left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right) \\ &= \sqrt{2}\sin\theta + \frac{2}{\sqrt{2}}\sin\theta + \frac{2}{\sqrt{2}}\cos\theta \\ &= 2\sqrt{2}\sin\theta + \sqrt{2}\cos\theta\end{aligned}$$

$P(2\sqrt{2}, \sqrt{2})$ とすると，$r = \sqrt{(2\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{10}$ だから

$$\begin{aligned}\sqrt{2}\sin\theta + 2\sin(\theta + 45^\circ) &= 2\sqrt{2}\sin\theta + \sqrt{2}\cos\theta \\ &= \boldsymbol{\sqrt{10}\sin(\theta+\alpha)} \\ &\left(\text{ただし，}\boldsymbol{\sin\alpha = \frac{1}{\sqrt{5}},\ \cos\alpha = \frac{2}{\sqrt{5}}}\right)\end{aligned}$$ (答)

【2】(1) $\sin\theta + \sqrt{3}\cos\theta = 2\sin\left(\theta + \frac{\pi}{3}\right)$

よって

$$2\sin\left(\theta + \frac{\pi}{3}\right) = 1$$
$$\sin\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}$$

ここで，$\frac{\pi}{3} \leqq \theta + \frac{\pi}{3} < \frac{7}{3}\pi$ より，$\theta + \frac{\pi}{3} = \frac{5}{6}\pi,\ \frac{13}{6}\pi$

したがって，$\theta = \boldsymbol{\frac{\pi}{2},\ \frac{11}{6}\pi}$ (答)

(2)

$$\begin{aligned}&\sin\left(\theta - \frac{5}{6}\pi\right) + \cos\theta \\ &= \sin\theta\cos\frac{5}{6}\pi - \cos\theta\sin\frac{5}{6}\pi + \cos\theta = -\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta + \cos\theta \\ &= -\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = \sin\left(\theta + \frac{5}{6}\pi\right)\end{aligned}$$

よって，$\sin\left(\theta + \frac{5}{6}\pi\right) = 0$

ここで，$\frac{5}{6}\pi \leqq \theta + \frac{5}{6}\pi < \frac{17}{6}\pi$ より，$\theta + \frac{5}{6}\pi = \pi,\ 2\pi$

したがって，$\theta = \boldsymbol{\frac{\pi}{6},\ \frac{7}{6}\pi}$ (答)

(3)

$$\begin{aligned}\sin^3\theta + \cos^3\theta &= (\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta) \\ &= (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)\end{aligned}$$

よって，$(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) = 0$

ここで，$\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

また，$1 - \sin\theta\cos\theta = 1 - \frac{1}{2}\sin 2\theta$ だから，$-1 \leqq \sin 2\theta \leqq 1$ より，

$\frac{1}{2} \leqq 1 - \sin\theta\cos\theta \leqq \frac{3}{2}$ だから，$1 - \sin\theta\cos\theta \fallingdotseq 0$

ゆえに，$\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = 0$ より，$\sin\left(\theta + \frac{\pi}{4}\right) = 0$

$\frac{\pi}{4} \leqq \theta + \frac{\pi}{4} < \frac{9}{4}\pi$ より，$\theta + \frac{\pi}{4} = \pi,\ 2\pi$

したがって，$\theta = \boldsymbol{\frac{3}{4}\pi,\ \frac{7}{4}\pi}$ (答)

(4)　積→和の公式より

$$\sin\left(\theta+\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{6}\right)=1$$
$$\frac{1}{2}\left\{\sin\left\{\left(\theta+\frac{\pi}{3}\right)+\left(\theta-\frac{\pi}{6}\right)\right\}+\sin\left\{\left(\theta+\frac{\pi}{3}\right)-\left(\theta-\frac{\pi}{6}\right)\right\}\right\}=1$$
$$\frac{1}{2}\left\{\sin\left(2\theta+\frac{\pi}{6}\right)+1\right\}=1$$
$$\sin\left(2\theta+\frac{\pi}{6}\right)=1$$

ここで，$\frac{\pi}{6}\leqq 2\theta+\frac{\pi}{6}<\frac{25}{6}\pi$ より，$2\theta+\frac{\pi}{6}=\frac{\pi}{2},\frac{5}{2}\pi$

よって，$\theta=\boldsymbol{\frac{\pi}{6},\frac{7}{6}\pi}$　　(答)

＜別解＞

角を $\theta-\frac{\pi}{6}$ に統一すると

$$\sin\left(\theta+\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{6}\right)=1$$
$$\sin\left\{\left(\theta-\frac{\pi}{6}\right)+\frac{\pi}{2}\right\}\cos\left(\theta-\frac{\pi}{6}\right)=1$$
$$\cos^2\left(\theta-\frac{\pi}{6}\right)=1$$
$$\cos\left(\theta-\frac{\pi}{6}\right)=\pm1$$

ここで，$-\frac{\pi}{6}\leqq\theta-\frac{\pi}{6}<\frac{11}{6}\pi$ より，$\theta-\frac{\pi}{6}=0,\pi$

よって，$\theta=\boldsymbol{\frac{\pi}{6},\frac{7}{6}\pi}$　　(答)

【3】(1)

$$\begin{aligned}\sin x+\sin 2x+\sin 3x&=0\\(\sin x+\sin 3x)+\sin 2x&=0\\2\sin 2x\cos(-x)+\sin 2x&=0\\2\sin 2x\cos x+\sin 2x&=0\\\sin 2x(2\cos x+1)&=0\end{aligned}$$

$\sin 2x=0$ のとき，$0\leqq 2x\leqq 2\pi$ より，$2x=0,\ \pi,\ 2\pi$ だから，
$x=0,\ \frac{\pi}{2},\ \pi$

$2\cos x+1=0$ のとき，$\cos x=-\frac{1}{2}$

よって，$x=\frac{2}{3}\pi$

以上より，$x=\boldsymbol{0,\ \frac{\pi}{2},\ \frac{2}{3}\pi,\ \pi}$　　(答)

＜別解＞

2 倍角，3 倍角の公式を用いて

$$\begin{aligned}\sin x + 2\sin x\cos x + (3\sin x - 4\sin^3 x) &= 0\\ 4\sin^3 x - 2\sin x\cos x - 4\sin x &= 0\\ 2\sin x(2\sin^2 x - \cos x - 2) &= 0\\ 2\sin x\{2(1-\cos^2 x) - \cos x - 2\} &= 0\\ 2\sin x(-2\cos^2 x - \cos x) &= 0\\ -2\sin x\cos x(2\cos x + 1) &= 0\end{aligned}$$

よって

$\sin x = 0$ より，$x = 0,\ \pi$
$\cos x = 0$ より，$x = \dfrac{\pi}{2}$
$\cos x = -\dfrac{1}{2}$ より，$x = \dfrac{2}{3}\pi$

以上より，$x = \boldsymbol{0,\ \dfrac{\pi}{2},\ \dfrac{2}{3}\pi,\ \pi}$　　(答)

(2)　半角の公式，2 倍角の公式より

$$\begin{aligned}\cos^2\theta + \sqrt{3}\sin\theta\cos\theta &= 1\\ \frac{1+\cos 2\theta}{2} + \sqrt{3}\times\frac{\sin 2\theta}{2} &= 1\\ \sqrt{3}\sin 2\theta + \cos 2\theta &= 1\\ 2\sin\left(2\theta + \frac{\pi}{6}\right) &= 1\\ \sin\left(2\theta + \frac{\pi}{6}\right) &= \frac{1}{2}\end{aligned}$$

$0 \leqq \theta < 2\pi$ より，$\dfrac{\pi}{6} \leqq 2\theta + \dfrac{\pi}{6} < \dfrac{25}{6}\pi$
よって，$2\theta + \dfrac{\pi}{6} = \dfrac{\pi}{6},\ \dfrac{5}{6}\pi,\ \dfrac{13}{6}\pi,\ \dfrac{17}{6}\pi$ より

$\boldsymbol{\theta = 0,\ \dfrac{\pi}{3},\ \pi,\ \dfrac{4}{3}\pi}$　　(答)

＜別解＞

明らかに，$\cos\theta \fallingdotseq 0$ より，両辺を $\cos^2\theta$ で割ると

$$\begin{aligned}1 + \sqrt{3}\times\frac{\sin\theta}{\cos\theta} &= \frac{1}{\cos^2\theta}\\ 1 + \sqrt{3}\tan\theta &= \tan^2\theta + 1\\ \tan^2\theta - \sqrt{3}\tan\theta &= 0\\ \tan\theta(\tan\theta - \sqrt{3}) &= 0\end{aligned}$$

$\tan\theta = 0$ のとき，$\theta = 0,\ \pi$
$\tan\theta = \sqrt{3}$ のとき，$\theta = \dfrac{\pi}{3},\ \dfrac{4}{3}\pi$

以上より，$\theta = \boldsymbol{0,\ \dfrac{\pi}{3},\ \pi,\ \dfrac{4}{3}\pi}$　　(答)

【4】(1) $\sqrt{3}\sin\theta+\cos\theta>\sqrt{2}$

$$2\sin\left(\theta+\frac{\pi}{6}\right)>\sqrt{2}$$

$$\sin\left(\theta+\frac{\pi}{6}\right)>\frac{\sqrt{2}}{2}$$

$0\leqq\theta<2\pi$ より，$\dfrac{\pi}{6}\leqq\theta+\dfrac{\pi}{6}<\dfrac{13}{6}\pi$

よって，$\dfrac{\pi}{4}<\theta+\dfrac{\pi}{6}<\dfrac{3}{4}\pi$

したがって，$\boldsymbol{\dfrac{\pi}{12}<\theta<\dfrac{7}{12}\pi}$ (答)

(2) $\sin\left(\dfrac{3}{2}\pi-\theta\right)=\sin\left\{(\pi-\theta)+\dfrac{\pi}{2}\right\}$

$$=\cos(\pi-\theta)$$

$$=-\cos\theta$$

よって

$$\sin\theta+\sqrt{3}\sin\left(\frac{3}{2}\pi-\theta\right)\geqq 1$$

$$\sin\theta-\sqrt{3}\cos\theta\geqq 1$$

$$2\sin\left(\theta+\frac{5}{3}\pi\right)\geqq 1$$

$$\sin\left(\theta+\frac{5}{3}\pi\right)\geqq\frac{1}{2}$$

$\dfrac{5}{3}\pi\leqq\theta+\dfrac{5}{3}\pi<\dfrac{11}{3}\pi$ より

$\dfrac{13}{6}\pi\leqq\theta+\dfrac{5}{3}\pi\leqq\dfrac{17}{6}\pi$

したがって，$\boldsymbol{\dfrac{\pi}{2}\leqq\theta\leqq\dfrac{7}{6}\pi}$ (答)

(3) 3倍角の公式より

$2\cos x+\cos 3x>0$

$2\cos x+4\cos^3 x-3\cos x>0$

$4\cos^3 x-\cos x>0$

$\cos x(2\cos x+1)(2\cos x-1)>0$

$\cos x=0$ では与式は不成立であるから，$\cos x\neq 0$ である．

(i) $\cos x>0$ のとき

つまり，$0\leqq x<\dfrac{\pi}{2}$ のとき，$2\cos x+1>0$ より

$2\cos x-1>0$

$\cos x>\dfrac{1}{2}$

$\therefore 0\leqq x<\dfrac{\pi}{3}$

(ii) $\cos x < 0$ のとき

つまり，$\dfrac{\pi}{2} < x < \pi$ のとき，$2\cos x - 1 < 0$ より

$$2\cos x + 1 > 0$$

$$\cos x > -\frac{1}{2}$$

$$\therefore \frac{\pi}{2} < x < \frac{2}{3}\pi$$

よって，$\boldsymbol{0 \leqq x < \dfrac{\pi}{3}},\ \boldsymbol{\dfrac{\pi}{2} < x < \dfrac{2}{3}\pi}$　　(答)

(4)
$$\cos^2\theta - 3\cos\theta - \sin^2\theta + 3\sin\theta \geqq 0$$
$$(\cos^2\theta - \sin^2\theta) - 3(\cos\theta - \sin\theta) \geqq 0$$
$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) - 3(\cos\theta - \sin\theta) \geqq 0$$
$$(\cos\theta - \sin\theta)(\cos\theta + \sin\theta - 3) \geqq 0$$

ここで

$$\cos\theta + \sin\theta - 3 = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) - 3 \leqq \sqrt{2} - 3 < 0$$

よって，$\cos\theta - \sin\theta \leqq 0$ より

$$\sqrt{2}\sin\left(\theta + \frac{3}{4}\pi\right) \leqq 0$$

ゆえに，$\sin\left(\theta + \dfrac{3}{4}\pi\right) \leqq 0$ であり，$\dfrac{3}{4}\pi \leqq \theta + \dfrac{3}{4}\pi < \dfrac{11}{4}\pi$ であることより，

$$\pi \leqq \theta + \frac{3}{4}\pi \leqq 2\pi$$

$\therefore \boldsymbol{\dfrac{\pi}{4} \leqq \theta \leqq \dfrac{5}{4}\pi}$　　(答)

【5】(1)
$$\begin{aligned} f(x) &= 2\sin x + 2(1 - \sin^2 x) + 1 \\ &= -2\sin^2 x + 2\sin x + 3 \\ &= -2\left(\sin x - \frac{1}{2}\right)^2 + \frac{7}{2} \end{aligned}$$

$\sin x = t$ とおくと，$0 \leqq x < 2\pi$ より

$$-1 \leqq t \leqq 1$$

よって題意は

$$f(x) = g(t) = -2\left(t - \frac{1}{2}\right)^2 + \frac{7}{2} \quad (-1 \leqq t \leqq 1)$$

の最大値と最小値を求めることに等しい．したがって，

$$\begin{cases} t = \dfrac{1}{2} \text{ のとき 最大値 } g\left(\dfrac{1}{2}\right) = \dfrac{7}{2} \\ t = -1 \text{ のとき 最小値 } g(-1) = -1 \end{cases}$$

ここで，$t = \dfrac{1}{2}$ となるのは

$$\sin x = \frac{1}{2}\ (\text{ただし，} 0 \leqq x < 2\pi) \text{ より}$$

$$x = \frac{\pi}{6}, \frac{5}{6}\pi \text{ のとき}$$

$t=-1$ となるのは

$$\sin x=-1\ (\text{ただし，}0\leqq x<2\pi)\ \text{より}$$
$$x=\frac{3}{2}\pi\ \text{のとき}$$

よって，

$$\begin{cases} x=\dfrac{\pi}{6},\dfrac{5}{6}\pi\ \text{のとき}\quad \text{最大値}\ \boldsymbol{\dfrac{7}{2}} & (\text{答}) \\ x=\dfrac{3}{2}\pi\ \text{のとき}\qquad \text{最小値}\ \boldsymbol{-1} & (\text{答}) \end{cases}$$

(2)
$$\begin{aligned} f(x)&=\sin^2 x+2\sin x\cos x+3\cos^2 x\\ &=(\sin^2 x+\cos^2 x)+2\sin x\cos x+(2\cos^2 x-1)+1\\ &=1+\sin 2x+\cos 2x+1\\ &=\sin 2x+\cos 2x+2 \end{aligned}$$

$t=\sin 2x+\cos 2x$ とすると $t=\sqrt{2}\sin\left(2x+\dfrac{\pi}{4}\right)$．ここで，$0\leqq x<2\pi$ より，$\dfrac{\pi}{4}\leqq 2x+\dfrac{\pi}{4}<\dfrac{17}{4}\pi$ であるから

$$-\sqrt{2}\leqq t\leqq\sqrt{2}$$

よって，題意は

$$f(x)=g(t)=t+2\qquad(-\sqrt{2}\leqq t\leqq\sqrt{2})$$

の最大値・最小値を求めることに等しい．したがって，

$$\begin{cases} t=\sqrt{2}\ \text{のとき 最大値}\ g(\sqrt{2})=2+\sqrt{2}\\ t=-\sqrt{2}\ \text{のとき 最小値}\ g(-\sqrt{2})=2-\sqrt{2} \end{cases}$$

ここで，$t=\sqrt{2}$ となるのは

$$\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)=\sqrt{2}\quad\left(\text{ただし，}\frac{\pi}{4}\leqq 2x+\frac{\pi}{4}<\frac{17}{4}\pi\right)\text{より}$$
$$2x+\frac{\pi}{4}=\frac{\pi}{2},\frac{5}{2}\pi\iff x=\frac{\pi}{8},\frac{9}{8}\pi\ \text{のとき}$$

$t=-\sqrt{2}$ となるのは

$$\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)=-\sqrt{2}\quad\left(\text{ただし，}\frac{\pi}{4}\leqq 2x+\frac{\pi}{4}<\frac{17}{4}\pi\right)\text{より}$$
$$2x+\frac{\pi}{4}=\frac{3}{2}\pi,\frac{7}{2}\pi\iff x=\frac{5}{8}\pi,\frac{13}{8}\pi\ \text{のとき}$$

よって

$$\begin{cases} x=\dfrac{\pi}{8},\dfrac{9}{8}\pi\ \text{のとき 最大値}\ \boldsymbol{2+\sqrt{2}} & (\text{答})\\ x=\dfrac{5}{8}\pi,\dfrac{13}{8}\pi\ \text{のとき 最小値}\ \boldsymbol{2-\sqrt{2}} & (\text{答}) \end{cases}$$

(3)
$$f(x)=(\sin x+\cos x)^2-2(\sin x+\cos x)$$

$t=\sin x+\cos x$ とおくと，$t=\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)$　ここで，$0\leqq x<2\pi$ より，$\dfrac{\pi}{4}\leqq x+\dfrac{\pi}{4}<\dfrac{9}{4}\pi$ であるから

$$-\sqrt{2}\leqq t\leqq\sqrt{2}$$

よって，題意は

$$\begin{aligned} f(x) = g(t) &= t^2 - 2t \\ &= (t-1)^2 - 1 \qquad (-\sqrt{2} \leqq t \leqq \sqrt{2}) \end{aligned}$$

の最大値・最小値を求めることに等しい．したがって，

$$\begin{cases} t = -\sqrt{2} \text{ のとき 最大値 } g(-\sqrt{2}) = 2 + 2\sqrt{2} \\ t = 1 \text{ のとき 最小値 } g(1) = -1 \end{cases}$$

ここで，$t = -\sqrt{2}$ となるのは

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = -\sqrt{2} \ \left(\text{ただし，} \frac{\pi}{4} \leqq x + \frac{\pi}{4} < \frac{9}{4}\pi\right) \text{より}$$
$$x + \frac{\pi}{4} = \frac{3}{2}\pi \iff x = \frac{5}{4}\pi \text{ のとき}$$

$t = 1$ となるのは

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1 \ \left(\text{ただし，} \frac{\pi}{4} \leqq x + \frac{\pi}{4} < \frac{9}{4}\pi\right) \text{ より}$$
$$x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3}{4}\pi \iff x = 0, \frac{\pi}{2} \text{ のとき}$$

よって，

$$\begin{cases} x = \frac{5}{4}\pi \text{ のとき 最大値 } \boldsymbol{2 + 2\sqrt{2}} \qquad \text{(答)} \\ x = 0, \frac{\pi}{2} \text{ のとき 最小値 } \boldsymbol{-1} \qquad \text{(答)} \end{cases}$$

添削課題

【1】 点 $P(a, b)$ に対して，$OP = r$，動径 OP が x 軸正の向きとなす角を α とすると

$$a = \boxed{\boldsymbol{r\cos\alpha}}, \qquad b = \boxed{\boldsymbol{r\sin\alpha}} \qquad \cdots\cdots①$$

とおけるので，加法定理より

$$\begin{aligned} a\sin\theta + b\cos\theta &= \boxed{\boldsymbol{r\cos\alpha}}\sin\theta + \boxed{\boldsymbol{r\sin\alpha}}\cos\theta \\ &= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \\ &= r\left(\boxed{\boldsymbol{\sin\theta\cos\alpha}} + \boxed{\boldsymbol{\cos\theta\sin\alpha}}\right) \\ &= r\boxed{\boldsymbol{\sin(\theta+\alpha)}} \end{aligned}$$

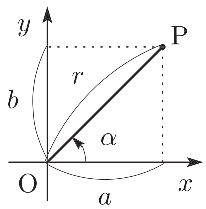
ここで

$$r = \boxed{\boldsymbol{\sqrt{a^2+b^2}}}$$

であるから，① より

$$\sin\alpha = \frac{b}{r} = \boxed{\boldsymbol{\frac{b}{\sqrt{a^2+b^2}}}},$$

$$\cos\alpha = \frac{a}{r} = \boxed{\boldsymbol{\frac{a}{\sqrt{a^2+b^2}}}}$$



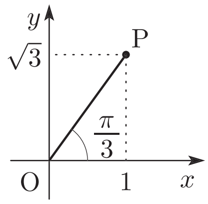
【2】 (1) $P(1, \sqrt{3})$ とすると

$$OP = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

OP と x 軸の正の向きとのなす角は，$\dfrac{\pi}{3}$

よって

$$\sin\theta + \sqrt{3}\cos\theta = \boldsymbol{2\sin\left(\theta + \frac{\pi}{3}\right)} \qquad (答)$$



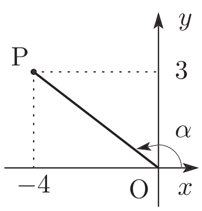
(2) $P(-4, 3)$ とすると

$$OP = \sqrt{(-4)^2 + 3^2} = 5$$

OP と x 軸の正の向きとのなす角を α とおくと

$$-4\sin\theta + 3\cos\theta = \boldsymbol{5\sin(\theta+\alpha)}$$

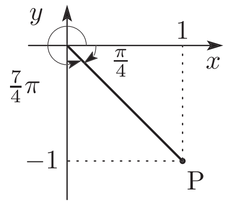
$$\left(ただし，\boldsymbol{\sin\alpha = \frac{3}{5}}，\boldsymbol{\cos\alpha = -\frac{4}{5}}\right) \qquad (答)$$



【3】P(1, −1) とすると

$$\mathrm{OP}=\sqrt{1^2+(-1)^2}=\sqrt{2}$$

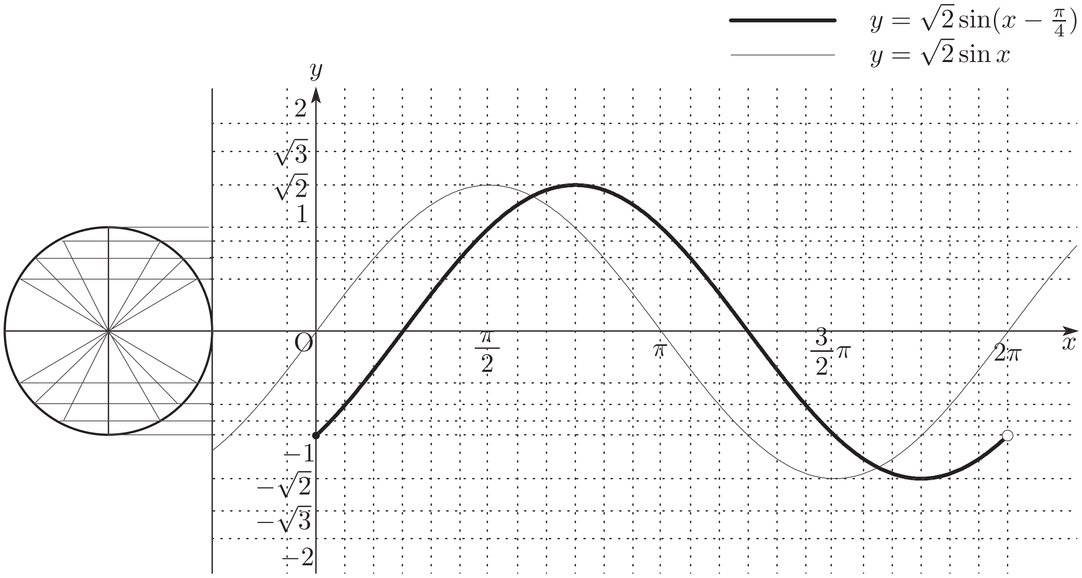
OP と x 軸の正の向きとのなす角は，$\dfrac{7}{4}\pi$



よって

$$y=\sin x-\cos x=\sqrt{2}\sin\left(x+\frac{7}{4}\pi\right)=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$$

したがって，$y=\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)$ のグラフは，$y=\sqrt{2}\sin x$ のグラフを x 軸正の方向に $\dfrac{\pi}{4}$ だけ平行移動したグラフだから，次のようになる．



(答)

【4】(1) 三角関数の合成を考え，

$$-\sin\theta+\sqrt{3}\cos\theta=1$$
$$\therefore 2\sin\left(\theta+\frac{2}{3}\pi\right)=1$$
$$\therefore \sin\left(\theta+\frac{2}{3}\pi\right)=\frac{1}{2}$$

$\frac{2}{3}\pi \leqq \theta+\frac{2}{3}\pi < \frac{8}{3}\pi$ より

$$\theta+\frac{2}{3}\pi=\frac{5}{6}\pi,\ \frac{13}{6}\pi$$
$$\therefore \boldsymbol{\theta=\frac{\pi}{6},\ \frac{3}{2}\pi} \qquad \text{(答)}$$

(2) 2 倍角の公式より

$$4\sin\theta\cos\theta-2\sin\theta-2\cos\theta+1=0$$

左辺を因数分解して

$$(2\sin\theta-1)(2\cos\theta-1)=0$$
$$\therefore \sin\theta=\frac{1}{2} \quad \text{または} \quad \cos\theta=\frac{1}{2}$$

$0\leqq\theta<2\pi$ より

$$\boldsymbol{\theta=\frac{\pi}{6},\ \frac{\pi}{3},\ \frac{5}{6}\pi,\ \frac{5}{3}\pi} \qquad \text{(答)}$$

(3) 半角，2 倍角の公式より

$$4\cdot\frac{1+\cos 2\theta}{2}-\sin 2\theta+2\cdot\frac{1-\cos 2\theta}{2}\leqq 2$$
$$3+\cos 2\theta-\sin 2\theta\leqq 2$$
$$\therefore \sin 2\theta-\cos 2\theta\geqq 1$$

さらに，三角関数の合成を用いると

$$\sqrt{2}\sin\left(2\theta-\frac{\pi}{4}\right)\geqq 1$$
$$\therefore \sin\left(2\theta-\frac{\pi}{4}\right)\geqq\frac{1}{\sqrt{2}}$$

ここで，$0\leqq\theta<\pi$ より，$-\frac{\pi}{4}\leqq 2\theta-\frac{\pi}{4}<\frac{7}{4}\pi$ であるから

$$\frac{\pi}{4}\leqq 2\theta-\frac{\pi}{4}\leqq\frac{3}{4}\pi$$
$$\therefore \boldsymbol{\frac{\pi}{4}\leqq\theta\leqq\frac{\pi}{2}} \qquad \text{(答)}$$

問題

【1】(1) $64 = 2^6$ より，$\pm\mathbf{2}$　　(答)

(2) $125 = 5^3$ より，$\mathbf{5}$　　(答)

(3) $-243 = (-3)^5$ より，$\mathbf{-3}$　　(答)

(4) $729 = (27)^2$ より，$\pm\mathbf{27}$　　(答)

(5) $729 = 9^3$ より，$\mathbf{9}$　　(答)

【2】(1) $\sqrt[3]{4}\sqrt[3]{6} = \sqrt[3]{24} = \sqrt[3]{2^3 \times 3} = \mathbf{2\sqrt[3]{3}}$　　(答)

(2) $\dfrac{\sqrt[3]{162}}{\sqrt[3]{6}} = \sqrt[3]{\dfrac{162}{6}} = \sqrt[3]{27} = \sqrt[3]{3^3} = 3^1 = \mathbf{3}$　　(答)

(3) $1024 = 2^{10}$ だから，

$$(\text{与式}) = \left\{(2^{10})^{\frac{1}{2}}\right\}^{\frac{1}{5}} = (2^{10})^{\frac{1}{10}} = \mathbf{2} \quad (\text{答})$$

(4) $\left(\dfrac{49}{64}\right)^{-1.5} = \left\{\left(\dfrac{7}{8}\right)^2\right\}^{-\frac{3}{2}} = \left(\dfrac{7}{8}\right)^{-3} = \mathbf{\dfrac{512}{343}}$　　(答)

(5)
$$\begin{aligned} 4^{-\frac{3}{2}} \times 27^{\frac{1}{3}} \div \sqrt{16^{-3}} &= \left(2^2\right)^{-\frac{3}{2}} \times \left(3^3\right)^{\frac{1}{3}} \div \left\{\left(2^4\right)^{-3}\right\}^{\frac{1}{2}} \\ &= 2^{-3} \times 3^1 \div 2^{-6} = 2^{-3-(-6)} \times 3^1 \\ &= \mathbf{24} \quad (\text{答}) \end{aligned}$$

(6) $\left(\sqrt[4]{5^3} \times \sqrt{27}\right)^{\frac{4}{3}} = \left(5^{\frac{3}{4}} \times \left(3^3\right)^{\frac{1}{2}}\right)^{\frac{4}{3}} = 5^{\frac{3}{4}\times\frac{4}{3}} \times 3^{\frac{3}{2}\times\frac{4}{3}} = 5 \times 3^2 = \mathbf{45}$　　(答)

(7)
$$\begin{aligned} &2\sqrt{3} \times \sqrt[3]{24} \div \sqrt{6} \div \sqrt[6]{72} \\ =&\left(2^1 \times 3^{\frac{1}{2}}\right) \times \left(2^3 \times 3\right)^{\frac{1}{3}} \times (2 \times 3)^{-\frac{1}{2}} \times \left(2^3 \times 3^2\right)^{-\frac{1}{6}} \\ =&\left(2^1 \times 3^{\frac{1}{2}}\right) \times \left(2^1 \times 3^{\frac{1}{3}}\right) \times \left(2^{-\frac{1}{2}} \times 3^{-\frac{1}{2}}\right) \times \left(2^{-\frac{1}{2}} \times 3^{-\frac{1}{3}}\right) \\ =&2^{1+1-\frac{1}{2}-\frac{1}{2}} \times 3^{\frac{1}{2}+\frac{1}{3}-\frac{1}{2}-\frac{1}{3}} = 2^1 \times 3^0 = \mathbf{2} \quad (\text{答}) \end{aligned}$$

(8)
$$\begin{aligned} &\sqrt[3]{-54} \div \sqrt[3]{\sqrt[2]{128}} \div \sqrt[3]{-4} \\ =&-\sqrt[3]{54} \times \frac{1}{\sqrt[6]{128}} \times \frac{1}{-\sqrt[3]{4}} = \sqrt[3]{\frac{54}{4}} \times \frac{1}{\sqrt[6]{128}} = \sqrt[3]{\frac{27}{2}} \times \frac{1}{\sqrt[6]{128}} \\ =&\frac{\sqrt[3]{3^3}}{\sqrt[3]{2}} \times \frac{1}{\sqrt[6]{2^7}} = 3 \times 2^{-\frac{1}{3}} \times 2^{-\frac{7}{6}} = 3 \times 2^{-\frac{1}{3}-\frac{7}{6}} = 3 \times 2^{-\frac{3}{2}} \\ =&\frac{3}{2\sqrt{2}} = \mathbf{\frac{3\sqrt{2}}{4}} \quad (\text{答}) \end{aligned}$$

(9) $81 = 3^4$ だから，

$$(\text{与式}) = (3^4)^{-\frac{3}{4}} = 3^{-3} = \mathbf{\frac{1}{27}} \quad (\text{答})$$

(10) $(\sqrt[4]{5}-\sqrt[4]{3})(\sqrt[4]{5}+\sqrt[4]{3})(\sqrt{5}+\sqrt{3})$

$=(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$

$=5-3=\mathbf{2}$ (答)

(11) $\left(5^{\frac{1}{2}}+5^{-\frac{1}{2}}\right)\left(5^{\frac{1}{2}}-5^{-\frac{1}{2}}\right)=\left(5^{\frac{1}{2}}\right)^2-\left(5^{-\frac{1}{2}}\right)^2$

$=5^1-5^{-1}=5-\dfrac{1}{5}=\mathbf{\dfrac{24}{5}}$ (答)

(12) (与式) $=(\sqrt[3]{4})^3-(\sqrt[3]{3})^3=4-3=\mathbf{1}$ (答)

【3】(1) $(8\sqrt{x})^{\frac{2}{3}}\times\sqrt[3]{x^2}=\left(2^3\times x^{\frac{1}{2}}\right)^{\frac{2}{3}}\times x^{\frac{2}{3}}=(2^3)^{\frac{2}{3}}\times\left(x^{\frac{1}{2}}\right)^{\frac{2}{3}}\times x^{\frac{2}{3}}$

$=2^{3\times\frac{2}{3}}\times x^{\frac{1}{2}\times\frac{2}{3}+\frac{2}{3}}=2^2\times x^1=\boldsymbol{4x}$ (答)

(2) $(64x^3y^{-9})^{\frac{1}{3}}=(2^6\times x^3\times y^{-9})^{\frac{1}{3}}$

$=(2^6)^{\frac{1}{3}}\times(x^3)^{\frac{1}{3}}\times(y^{-9})^{\frac{1}{3}}$

$=2^2\times x^1\times y^{-3}=\boldsymbol{\dfrac{4x}{y^3}}$ (答)

(3) (与式) $=x^{\frac{4}{3}-\frac{2}{3}+\frac{1}{3}}y^{-\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}=x^1y^0=\boldsymbol{x}$ (答)

(4) $\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)^2=\left(x^{\frac{1}{2}}\right)^2+2x^{\frac{1}{2}}x^{-\frac{1}{2}}+\left(x^{-\frac{1}{2}}\right)^2=x^1+2x^0+x^{-1}$

$=\boldsymbol{x+2+\dfrac{1}{x}}$ (答)

<参考>

(与式) $=\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2=\boldsymbol{x+2+\dfrac{1}{x}}$ (答)

(5) (与式) $=(x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}+1)=\boldsymbol{x-1}$ (答)

(6) $(x-y)\div\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)=\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)\div\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)=\boldsymbol{\sqrt{x}+\sqrt{y}}$ (答)

(7) $\left(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}\right)$

$=\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}\right)$

$=\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^2-\left(x^{\frac{1}{4}}y^{\frac{1}{4}}\right)^2=x^1+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y^1-x^{\frac{1}{2}}y^{\frac{1}{2}}$

$=\boldsymbol{x+\sqrt{xy}+y}$ (答)

(8) $\left(x^{\frac{a}{3}}+y^{-\frac{a}{3}}\right)\left\{x^{\frac{2a}{3}}-(xy^{-1})^{\frac{a}{3}}+y^{-\frac{2a}{3}}\right\}=\left(x^{\frac{a}{3}}\right)^3+\left(y^{-\frac{a}{3}}\right)^3=x^a+y^{-a}$

$=\boldsymbol{x^a+\dfrac{1}{y^a}}$ (答)

(9) $(xy^{-3}z^3)^{\frac{1}{2}}\times(x^7y^4z^2)^{\frac{1}{3}}\times(x^{-5}yz)^{\frac{1}{6}}=x^{\frac{1}{2}+\frac{7}{3}-\frac{5}{6}}y^{-\frac{3}{2}+\frac{4}{3}+\frac{1}{6}}z^{\frac{3}{2}+\frac{2}{3}+\frac{1}{6}}=x^2y^0z^{\frac{7}{3}}$

$=\boldsymbol{x^2z^{\frac{7}{3}}}$ (答)

(10) $\left(x^{\frac{a}{a-b}}\right)^{\frac{1}{c-a}}\left(x^{\frac{b}{b-c}}\right)^{\frac{1}{a-b}}\left(x^{\frac{c}{c-a}}\right)^{\frac{1}{b-c}}=x^{\frac{a(b-c)+b(c-a)+c(a-b)}{(a-b)(b-c)(c-a)}}=x^0=\mathbf{1}$ (答)

(11) (与式) $=(x^3-y^{-3})\div(x^2+xy^{-1}+y^{-2})$

$=(x-y^{-1})(x^2+xy^{-1}+y^{-2})\div(x^2+xy^{-1}+y^{-2})$

$=\boldsymbol{x-\dfrac{1}{y}}$ (答)

【4】(1) $\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)^2=9$ より，

$$x+x^{-1}+2=9$$

よって，

$$x+x^{-1}=\mathbf{7}\qquad\textbf{(答)}$$

(2)

$$\begin{aligned}x^{\frac{3}{2}}+x^{-\frac{3}{2}}&=\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)\left\{\left(x^{\frac{1}{2}}\right)^2-x^{\frac{1}{2}}\cdot x^{-\frac{1}{2}}+\left(x^{-\frac{1}{2}}\right)^2\right\}\\&=\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)\left(x+x^{-1}-1\right)\\&=3\times(7-1)=\mathbf{18}\qquad\textbf{(答)}\end{aligned}$$

【5】$a^x-a^{-x}=0$ のとき，両辺に a^x をかけて，

$$a^{2x}-1=0\qquad\therefore\quad a^{2x}=1$$

となり，(題意に) 矛盾するので，

$$a^x-a^{-x}\fallingdotseq 0$$

$$\begin{aligned}\frac{a^{3x}-a^{-3x}}{a^x-a^{-x}}&=\frac{(a^x-a^{-x})\left\{(a^x)^2+a^xa^{-x}+(a^{-x})^2\right\}}{a^x-a^{-x}}=a^{2x}+\left(a^{2x}\right)^{-1}+1\\&=4+4^{-1}+1=4+\frac{1}{4}+1=\mathbf{\frac{21}{4}}\qquad\textbf{(答)}\end{aligned}$$

＜別解＞
$a^{2x}=4$ より，$(a^x)^2=4$.
$a^x>0$ より，$a^x=2$.
よって，

$$(\text{求値式})=\frac{(a^x)^3-(a^x)^{-3}}{a^x-(a^x)^{-1}}=\frac{2^3-2^{-3}}{2-2^{-1}}=\frac{8-\dfrac{1}{8}}{2-\dfrac{1}{2}}=\mathbf{\frac{21}{4}}\qquad(\text{答})$$

【6】$x=\dfrac{1}{2}\left(5^{\frac{1}{n}}-5^{-\frac{1}{n}}\right)$ のとき，

$$1+x^2=1+\left(\frac{1}{2}\left(5^{\frac{1}{n}}-5^{-\frac{1}{n}}\right)\right)^2=\left(\frac{1}{2}\left(5^{\frac{1}{n}}+5^{-\frac{1}{n}}\right)\right)^2$$

であるから，

$$\begin{aligned}(\text{求値式})&=\left\{\frac{1}{2}\left(5^{\frac{1}{n}}-5^{-\frac{1}{n}}\right)+\sqrt{\left(\frac{1}{2}\left(5^{\frac{1}{n}}+5^{-\frac{1}{n}}\right)\right)^2}\right\}^n\\&=\left\{\frac{1}{2}\left(5^{\frac{1}{n}}-5^{-\frac{1}{n}}\right)+\frac{1}{2}\left(5^{\frac{1}{n}}+5^{-\frac{1}{n}}\right)\right\}^n\\&=(5^{\frac{1}{n}})^n=\mathbf{5}\qquad\textbf{(答)}\end{aligned}$$

添削課題

【1】(1)

$$2^3 \times 2^0 \times (2^{-1})^2 = 8 \times 1 \times \left(\frac{1}{2}\right)^2$$
$$= \mathbf{2} \qquad (答)$$

(2)

$$a^4 \div a^{-2} \times a^0 = a^4 \div \frac{1}{a^2} \times 1$$
$$= \boldsymbol{a^6} \qquad (答)$$

(3)

$$x^2 \div \left(\frac{1}{x}\right)^{-2} = x^2 \div \left(x^{-1}\right)^{-2}$$
$$= x^2 \div x^2$$
$$= \mathbf{1} \qquad (答)$$

(4)

$$36x^{-1}y^2 \div (-2x^3y^{-1})^2 = 36x^{-1}y^2 \div 4x^6y^{-2}$$
$$= \frac{36y^2}{x} \div \frac{4x^6}{y^2}$$
$$= \boldsymbol{\frac{9y^4}{x^7}} \qquad (答)$$

【2】(1) 2 乗して 25 になる実数であるから，$\pm\mathbf{5}$　　(答)

(2) $\sqrt{25} = \mathbf{5}$　　(答)

(3) 3 乗して -27 になる実数であるから，$\mathbf{-3}$　　(答)

(4) $\sqrt[3]{-27} = \mathbf{-3}$　　(答)

(5) 4 乗して 16 になる実数であるから，$\pm\mathbf{2}$　　(答)

(6) $\sqrt[4]{16} = \mathbf{2}$　　(答)

【3】(1)

$$\left(\sqrt[4]{3}\right)^8 = \sqrt[4]{3^8}$$
$$= \left(\sqrt[4]{3^2}\right)^4$$
$$= 3^2$$
$$= \mathbf{9} \qquad (答)$$

(2)

$$\sqrt[3]{32} \div \sqrt[3]{4} = \sqrt[3]{\frac{32}{4}}$$
$$= \sqrt[3]{8}$$
$$= \sqrt[3]{2^3}$$
$$= \left(\sqrt[3]{2}\right)^3$$
$$= \mathbf{2} \qquad (答)$$

(3)

$$\begin{aligned}\left(\frac{16}{81}\right)^{-0.75} &= \left(\frac{16}{81}\right)^{-\frac{3}{4}}\\ &= \left(\frac{81}{16}\right)^{\frac{3}{4}}\\ &= \left(\frac{3^4}{2^4}\right)^{\frac{3}{4}}\\ &= \frac{3^3}{2^3}\\ &= \mathbf{\frac{27}{8}} \qquad \text{(答)}\end{aligned}$$

(4)

$$\begin{aligned}4^{\frac{2}{3}} \div 40^{\frac{1}{3}} \times 50^{\frac{2}{3}} &= \left(2^2\right)^{\frac{2}{3}} \times \left(2^3 \cdot 5\right)^{-\frac{1}{3}} \times \left(2 \cdot 5^2\right)^{\frac{2}{3}}\\ &= 2^{2\times\frac{2}{3}+3\times\left(-\frac{1}{3}\right)+1\times\frac{2}{3}} \cdot 5^{-\frac{1}{3}+2\times\frac{2}{3}}\\ &= 2^1 \cdot 5^1\\ &= \mathbf{10} \qquad \text{(答)}\end{aligned}$$

【4】(1)

$$\begin{aligned}\sqrt{a}\sqrt[3]{a}\sqrt[4]{a} &= a^{\frac{1}{2}}a^{\frac{1}{3}}a^{\frac{1}{4}}\\ &= a^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}\\ &= \boldsymbol{a^{\frac{13}{12}}} \qquad \text{(答)}\end{aligned}$$

(2)

$$\begin{aligned}\sqrt[4]{a\sqrt[3]{a\sqrt{a}}} &= \left\{a \cdot \left(a \cdot a^{\frac{1}{2}}\right)^{\frac{1}{3}}\right\}^{\frac{1}{4}}\\ &= \left\{a \cdot \left(a^{\frac{3}{2}}\right)^{\frac{1}{3}}\right\}^{\frac{1}{4}}\\ &= a^{\left(1+\frac{3}{2}\times\frac{1}{3}\right)\times\frac{1}{4}}\\ &= \boldsymbol{a^{\frac{3}{8}}} \qquad \text{(答)}\end{aligned}$$

M1T

高１難関大数学

Z-KAI

会員番号		氏　名	