3MJSS/3MJS/3MJ

本科 2 期 12 月度

解答

Ｚ会東大進学教室

中３選抜東大・医学部数学

中３数学

中３東大数学

２４章　三角比（４）－三角方程式と三角不等式－

問題

【1】(1)

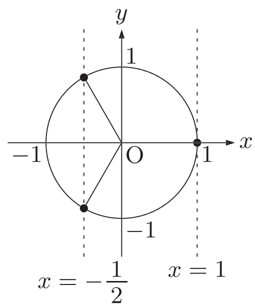
$$2\cos^2\theta - \cos\theta - 1 = 0$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$0^\circ \leqq \theta < 360^\circ$ より，$-1 \leqq \cos\theta \leqq 1$ なので，

$\cos\theta = -\dfrac{1}{2}$，1

よって，

$\boldsymbol{\theta = 0^\circ}$，$\boldsymbol{120^\circ}$，$\boldsymbol{240^\circ}$



(2)

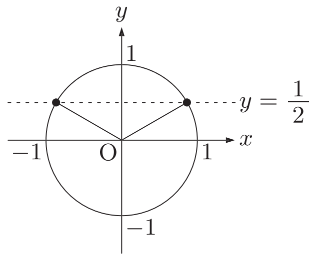
$$2\sin^2\theta - 5\sin\theta + 2 = 0$$
$$(2\sin\theta - 1)(\sin\theta - 2) = 0$$

$0^\circ \leqq \theta < 360^\circ$ より，$-1 \leqq \sin\theta \leqq 1$ なので，

$\sin\theta = \dfrac{1}{2}$

よって，

$\boldsymbol{\theta = 30^\circ}$，$\boldsymbol{150^\circ}$



(3)

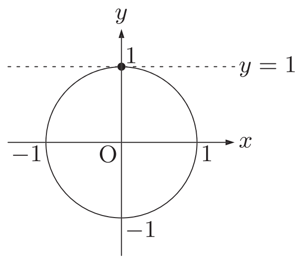
$$\cos^2\theta + 3\sin\theta - 3 = 0$$
$$(1 - \sin^2\theta) + 3\sin\theta - 3 = 0$$
$$-\sin^2\theta + 3\sin\theta - 2 = 0$$
$$\sin^2\theta - 3\sin\theta + 2 = 0$$
$$(\sin\theta - 1)(\sin\theta - 2) = 0$$

$0^\circ \leqq \theta < 360^\circ$ より，$-1 \leqq \sin\theta \leqq 1$ なので，

$\sin\theta = 1$

よって，

$\boldsymbol{\theta = 90^\circ}$



(4)

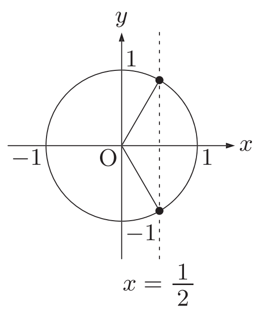
$$\frac{1}{4} + \cos\theta - \sin^2\theta = 0$$
$$\frac{1}{4} + \cos\theta - (1 - \cos^2\theta) = 0$$
$$4\cos^2\theta + 4\cos\theta - 3 = 0$$
$$(2\cos\theta + 3)(2\cos\theta - 1) = 0$$

$0^\circ \leqq \theta < 360^\circ$ より，$-1 \leqq \cos\theta \leqq 1$ なので，

$\cos\theta = \dfrac{1}{2}$

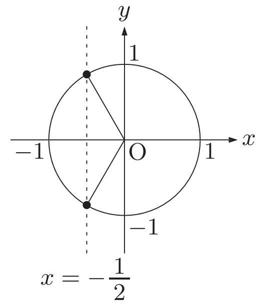
よって，

$\boldsymbol{\theta = 60^\circ}$，$\boldsymbol{300^\circ}$



(5)

$$2\sin^2\theta + 5\cos\theta + 1 = 0$$
$$2(1-\cos^2\theta) + 5\cos\theta + 1 = 0$$
$$-2\cos^2\theta + 5\cos\theta + 3 = 0$$
$$2\cos^2\theta - 5\cos\theta - 3 = 0$$
$$(2\cos\theta + 1)(\cos\theta - 3) = 0$$



$0° \leqq \theta < 360°$ より，$-1 \leqq \cos\theta \leqq 1$ なので，

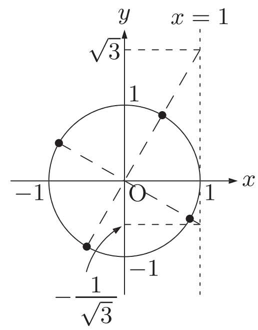
$$\cos\theta = -\frac{1}{2}$$

よって，

$$\boldsymbol{\theta = 120°,\ 240°}$$

(6)

$$\sqrt{3}\tan^2\theta = 2\tan\theta + \sqrt{3}$$
$$\sqrt{3}\tan^2\theta - 2\tan\theta - \sqrt{3} = 0$$
$$(\sqrt{3}\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$$
$$\tan\theta = -\frac{1}{\sqrt{3}},\ \sqrt{3}$$



$\tan\theta = -\dfrac{1}{\sqrt{3}}$ のとき，

$$\theta = 150°,\ 330°$$

$\tan\theta = \sqrt{3}$ のとき，

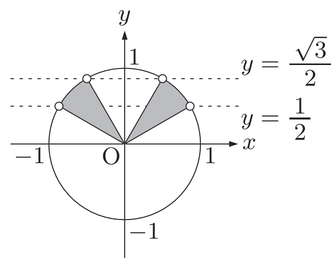
$$\theta = 60°,\ 240°$$

よって，

$$\boldsymbol{\theta = 60°,\ 150°,\ 240°,\ 330°}$$

【2】(1)

$$\frac{1}{2} < \sin\theta < \frac{\sqrt{3}}{2}$$



$\sin\theta = \dfrac{1}{2}$ のとき，$\theta = 30°,\ 150°$

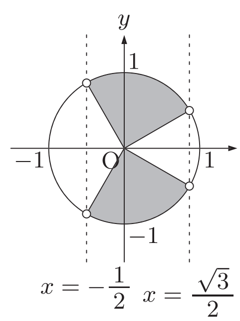
$\sin\theta = \dfrac{\sqrt{3}}{2}$ のとき，$\theta = 60°,\ 120°$

よって，

$$\boldsymbol{30° < \theta < 60°,\ 120° < \theta < 150°}$$

(2)

$$-\frac{1}{2} < \cos\theta < \frac{\sqrt{3}}{2}$$



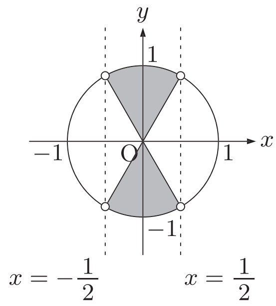
$\cos\theta = -\dfrac{1}{2}$ のとき，$\theta = 120°,\ 240°$

$\cos\theta = \dfrac{\sqrt{3}}{2}$ のとき，$\theta = 30°,\ 330°$

よって，

$$\boldsymbol{30° < \theta < 120°,\ 240° < \theta < 330°}$$

(3)
$$4\cos^2\theta - 1 < 0$$
$$(2\cos\theta + 1)(2\cos\theta - 1) < 0$$
$$-\frac{1}{2} < \cos\theta < \frac{1}{2}$$



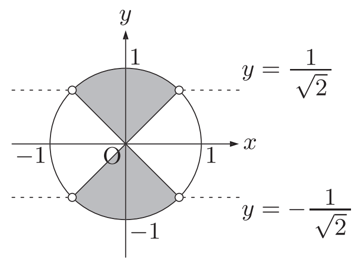
$\cos\theta = -\frac{1}{2}$ のとき，$\theta = 120°$，$240°$

$\cos\theta = \frac{1}{2}$ のとき，$\theta = 60°$，$300°$

よって，

$\boldsymbol{60° < \theta < 120°,\ 240° < \theta < 300°}$

(4)
$$2\sin^2\theta > 1$$
$$2\sin^2\theta - 1 > 0$$
$$(\sqrt{2}\sin\theta + 1)(\sqrt{2}\sin\theta - 1) > 0$$
$$\sin\theta < -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} < \sin\theta$$



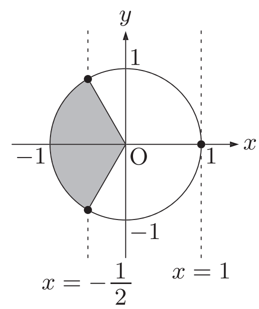
$\sin\theta = -\frac{1}{\sqrt{2}}$ のとき，$\theta = 225°$，$315°$

$\sin\theta = \frac{1}{\sqrt{2}}$ のとき，$\theta = 45°$，$135°$

よって，

$\boldsymbol{45° < \theta < 135°,\ 225° < \theta < 315°}$

(5)
$$2\cos^2\theta - \cos\theta - 1 \geqq 0$$
$$(2\cos\theta + 1)(\cos\theta - 1) \geqq 0$$
$$\cos\theta \leqq -\frac{1}{2},\ 1 \leqq \cos\theta$$



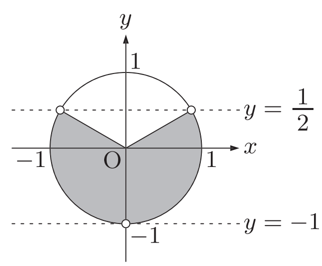
$\cos\theta = -\frac{1}{2}$ のとき，$\theta = 120°$，$240°$

$\cos\theta = 1$ のとき，$\theta = 0°$

よって，

$\boldsymbol{\theta = 0°,\ 120° \leqq \theta \leqq 240°}$

(6)
$$2\sin^2\theta + \sin\theta - 1 < 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) < 0$$
$$-1 < \sin\theta < \frac{1}{2}$$



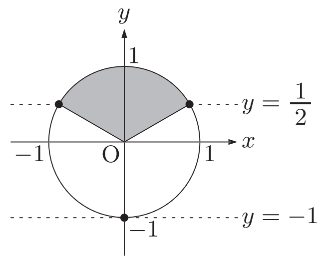
$\sin\theta = -1$ のとき，$\theta = 270°$

$\sin\theta = \frac{1}{2}$ のとき，$\theta = 30°$，$150°$

よって，

$\boldsymbol{0° \leqq \theta < 30°,\ 150° < \theta < 270°,\ 270° < \theta < 360°}$

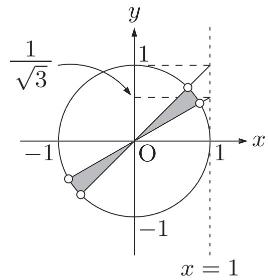
(7)
$$\begin{aligned} 2\cos^2\theta - \sin\theta - 1 &\leqq 0 \\ 2(1-\sin^2\theta) - \sin\theta - 1 &\leqq 0 \\ -2\sin^2\theta - \sin\theta + 1 &\leqq 0 \\ 2\sin^2\theta + \sin\theta - 1 &\geqq 0 \\ (2\sin\theta - 1)(\sin\theta + 1) &\geqq 0 \\ \sin\theta \leqq -1,\ \frac{1}{2} &\leqq \sin\theta \end{aligned}$$



$\sin\theta = -1$ のとき，$\theta = 270^\circ$
$\sin\theta = \frac{1}{2}$ のとき，$\theta = 30^\circ$，150°
よって，

$\boldsymbol{30^\circ \leqq \theta \leqq 150^\circ, \theta = 270^\circ}$

(8)
$$\begin{aligned} 3\tan^2\theta - (3+\sqrt{3})\tan\theta + \sqrt{3} &< 0 \\ (3\tan\theta - \sqrt{3})(\tan\theta - 1) &< 0 \\ \frac{1}{\sqrt{3}} < \tan\theta &< 1 \end{aligned}$$



$\tan\theta = \frac{1}{\sqrt{3}}$ のとき，$\theta = 30^\circ$，210°
$\tan\theta = 1$ のとき，$\theta = 45^\circ$，225°
よって，

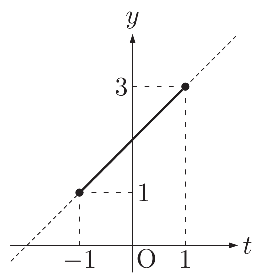
$\boldsymbol{30^\circ < \theta < 45^\circ,\ 210^\circ < \theta < 225^\circ}$

【3】(1) $\cos\theta = t$ とおく．$0^\circ \leqq \theta < 360^\circ$ から，

$-1 \leqq t \leqq 1$

また，

$$\begin{aligned} y &= \cos\theta + 2 \\ &= t + 2 \qquad (-1 \leqq t \leqq 1) \end{aligned}$$

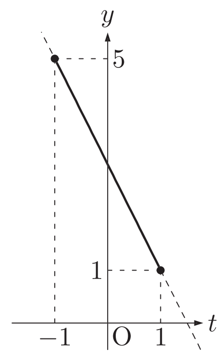


グラフから，$t = 1$ で最大値 3 をとり，
このとき，$\cos\theta = 1$ から，$\theta = 0^\circ$
$t = -1$ で最小値 1 をとり，
このとき，$\cos\theta = -1$ から，$\theta = 180^\circ$
よって，

最大値：3 ($\boldsymbol{\theta = 0^\circ}$)，最小値：1 ($\boldsymbol{\theta = 180^\circ}$)

(2) $\sin\theta = t$ とおく．$0^\circ \leqq \theta < 360^\circ$ から，

$$-1 \leqq t \leqq 1$$



また，

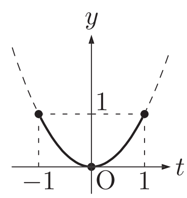
$$\begin{aligned} y &= 3 - 2\sin\theta \\ &= -2t + 3 \qquad (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = -1$ で最大値 5 をとり，
このとき，$\sin\theta = -1$ から，$\theta = 270^\circ$
$t = 1$ で最小値 1 をとり，
このとき，$\sin\theta = 1$ から，$\theta = 90^\circ$
よって，

最大値：5　($\boldsymbol{\theta = 270^\circ}$)，最小値：1　($\boldsymbol{\theta = 90^\circ}$)

(3) $\sin\theta = t$ とおく．$0^\circ \leqq \theta < 360^\circ$ から，

$$-1 \leqq t \leqq 1$$



また，

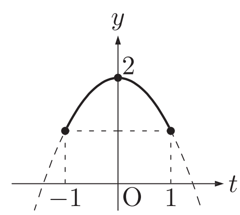
$$\begin{aligned} y &= \sin^2\theta \\ &= t^2 \quad (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = -1$，1 で最大値 1 をとり，
このとき，$\sin\theta = -1$，1 から，$\theta = 90^\circ$，270°
$t = 0$ で最小値 0 をとり，
このとき，$\sin\theta = 0$ から，$\theta = 0^\circ$，180°
よって，

最大値：1 ($\boldsymbol{\theta = 90^\circ}$，$\boldsymbol{270^\circ}$)，最小値：0 ($\boldsymbol{\theta = 0^\circ}$，$\boldsymbol{180^\circ}$)

(4) $\cos\theta = t$ とおく．$0^\circ \leqq \theta < 360^\circ$ から，

$$-1 \leqq t \leqq 1$$



また，

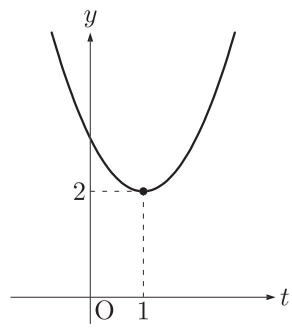
$$\begin{aligned} y &= 2 - \cos^2\theta \\ &= -t^2 + 2 \quad (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = 0$ で最大値 2 をとり，
このとき，$\cos\theta = 0$ から，$\theta = 90^\circ$，270°
$t = -1$，1 で最小値 1 をとり，
このとき，$\cos\theta = -1$，1 から，$\theta = 0^\circ$，180°
よって，

最大値：2 ($\boldsymbol{\theta = 90^\circ}$，$\boldsymbol{270^\circ}$)，最小値：1 ($\boldsymbol{\theta = 0^\circ}$，$\boldsymbol{180^\circ}$)

(5) $\tan\theta = t$ とおくと，$0° \leqq \theta < 360°$ から，
t は任意の実数なので，

$$\begin{aligned} y &= \tan^2\theta - 2\tan\theta + 3 \\ &= t^2 - 2t + 3 \\ &= (t-1)^2 + 2 \end{aligned}$$

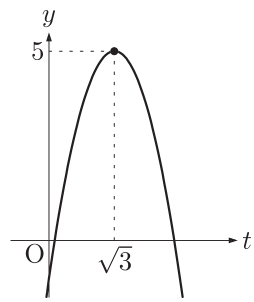


グラフから，$t = 1$ で最小値 2 をとり，
このとき，$\tan\theta = 1$ から，$\theta = 45°$，$225°$
よって，

最大値：なし，最小値：2 ($\boldsymbol{\theta = 45°}$，225°)

(6) $\tan\theta = t$ とおくと，$0° \leqq \theta < 360°$ から，
t は任意の実数なので，

$$\begin{aligned} y &= -2\tan^2\theta + 4\sqrt{3}\tan\theta - 1 \\ &= -2t^2 + 4\sqrt{3}t - 1 \\ &= -2(t^2 - 2\sqrt{3}t) - 1 \\ &= -2(t - \sqrt{3})^2 + 5 \end{aligned}$$



グラフから，$t = \sqrt{3}$ で最大値 5 をとり，
このとき，$\tan\theta = \sqrt{3}$ から，$\theta = 60°$，$240°$
よって，

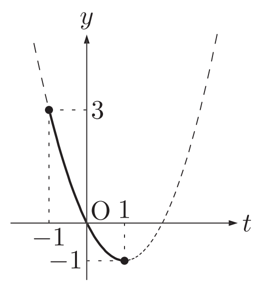
最大値：5 ($\boldsymbol{\theta = 60°}$，240°)，最小値：なし

(7) $\sin\theta = t$ とおく．$0° \leqq \theta < 360°$ から，

$$-1 \leqq t \leqq 1$$

また，

$$\begin{aligned} y &= \sin^2\theta - 2\sin\theta \\ &= t^2 - 2t \\ &= (t-1)^2 - 1 \ (-1 \leqq t \leqq 1) \end{aligned}$$



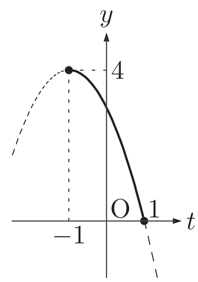
グラフから，$t = -1$ で最大値 3 をとり，
このとき，$\sin\theta = -1$ から，$\theta = 270°$
$t = 1$ で最小値 -1 をとり，
このとき，$\sin\theta = 1$ から，$\theta = 90°$
よって，

最大値：3 ($\boldsymbol{\theta = 270°}$)，最小値：$\boldsymbol{-1}$ ($\boldsymbol{\theta = 90°}$)

(8) $\cos\theta = t$ とおく．$0° \leqq \theta < 360°$ から，

$$-1 \leqq t \leqq 1$$

また，

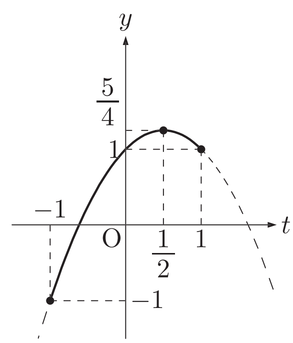


$$\begin{aligned} y &= -\cos^2\theta - 2\cos\theta + 3 \\ &= -t^2 - 2t + 3 \\ &= -(t^2 + 2t) + 3 \\ &= -(t+1)^2 + 4 \ (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = -1$ で最大値 4 をとり，
このとき，$\cos\theta = -1$ から，$\theta = 180°$
$t = 1$ で最小値 0 をとり，
このとき，$\cos\theta = 1$ から，$\theta = 0°$
よって，

最大値：4 ($\boldsymbol{\theta = 180°}$)，最小値 0 ($\boldsymbol{\theta = 0°}$)

(9)
$$\begin{aligned} y &= \cos^2\theta + \sin\theta \\ &= (1 - \sin^2\theta) + \sin\theta \\ &= -\sin^2\theta + \sin\theta + 1 \end{aligned}$$



より，$\sin\theta = t$ とおく．$0° \leqq \theta < 360°$ から，

$$-1 \leqq t \leqq 1$$

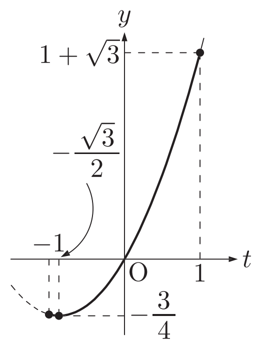
また，

$$\begin{aligned} y &= -\sin^2\theta + \sin\theta + 1 \\ &= -t^2 + t + 1 \\ &= -(t^2 - t) + 1 \\ &= -\left(t - \frac{1}{2}\right)^2 + \frac{5}{4} \ (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = \dfrac{1}{2}$ で最大値 $\dfrac{5}{4}$ をとり，
このとき，$\sin\theta = \dfrac{1}{2}$ から，$\theta = 30°，150°$
$t = -1$ で最小値 -1 をとり，
このとき，$\sin\theta = -1$ から，$\theta = 270°$
よって，

最大値：$\boldsymbol{\dfrac{5}{4}}$ ($\boldsymbol{\theta = 30°，150°}$)，最小値：$\boldsymbol{-1}$ ($\boldsymbol{\theta = 270°}$)

(10)
$$\begin{aligned} y &= -\sin^2\theta + \sqrt{3}\cos\theta + 1 \\ &= -(1-\cos^2\theta) + \sqrt{3}\cos\theta + 1 \\ &= \cos^2\theta + \sqrt{3}\cos\theta \end{aligned}$$



より，$\cos\theta = t$ とおく．$0^\circ \leqq \theta < 360^\circ$ から，

$$-1 \leqq t \leqq 1$$

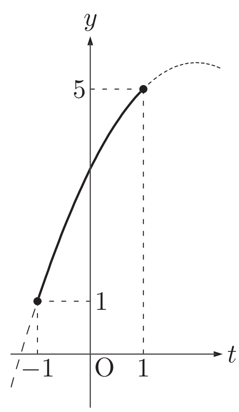
また，

$$\begin{aligned} y &= \cos^2\theta + \sqrt{3}\cos\theta \\ &= t^2 + \sqrt{3}t \\ &= \left(t + \frac{\sqrt{3}}{2}\right)^2 - \frac{3}{4} \ (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = 1$ で最大値 $1 + \sqrt{3}$ をとり，
このとき，$\cos\theta = 1$ から，$\theta = 0^\circ$
$t = -\dfrac{\sqrt{3}}{2}$ で最小値 $-\dfrac{3}{4}$ をとり，
このとき，$\cos\theta = -\dfrac{\sqrt{3}}{2}$ から，$\theta = 150^\circ$，210°
よって，

最大値：$1 + \sqrt{3}$ ($\theta = 0^\circ$)，最小値：$-\dfrac{3}{4}$ ($\theta = 150^\circ$，210°)

(11)
$$\begin{aligned} y &= \frac{1}{2}\sin^2\theta + 2\cos\theta + 3 \\ &= \frac{1}{2}(1-\cos^2\theta) + 2\cos\theta + 3 \\ &= -\frac{1}{2}\cos^2\theta + 2\cos\theta + \frac{7}{2} \end{aligned}$$



より，$\cos\theta = t$ とおく．$0^\circ \leqq \theta < 360^\circ$ から，

$$-1 \leqq t \leqq 1$$

また，

$$\begin{aligned} y &= -\frac{1}{2}\cos^2\theta + 2\cos\theta + \frac{7}{2} \\ &= -\frac{1}{2}t^2 + 2t + \frac{7}{2} \\ &= -\frac{1}{2}(t^2 - 4t) + \frac{7}{2} \\ &= -\frac{1}{2}(t-2)^2 + \frac{11}{2} \ (-1 \leqq t \leqq 1) \end{aligned}$$

グラフから，$t = 1$ で最大値 5 をとり，このとき，$\cos\theta = 1$ から，$\theta = 0^\circ$
$t = -1$ で最小値 1 をとり，このとき，$\cos\theta = -1$ から，$\theta = 180^\circ$
よって，

最大値：5 ($\theta = 0^\circ$)，最小値：1 ($\theta = 180^\circ$)

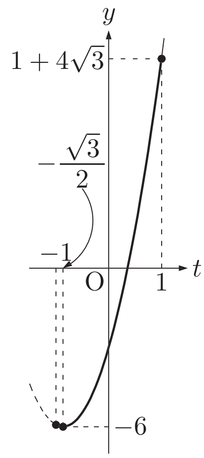
(12)
$$\begin{aligned} y &= \sin^2\theta - 3\cos^2\theta + 4\sqrt{3}\sin\theta \\ &= \sin^2\theta - 3(1-\sin^2\theta) + 4\sqrt{3}\sin\theta \\ &= 4\sin^2\theta + 4\sqrt{3}\sin\theta - 3 \end{aligned}$$

より，$\sin\theta = t$ とおく．$0° \leqq \theta < 360°$ から，

$$-1 \leqq t \leqq 1$$

また，

$$\begin{aligned} y &= 4\sin^2\theta + 4\sqrt{3}\sin\theta - 3 \\ &= 4t^2 + 4\sqrt{3}t - 3 \\ &= 4(t^2 + \sqrt{3}t) - 3 \\ &= 4\left(t + \frac{\sqrt{3}}{2}\right)^2 - 6 \quad (-1 \leqq t \leqq 1) \end{aligned}$$



グラフから，$t = 1$ で最大値 $1 + 4\sqrt{3}$ をとり，
このとき，$\sin\theta = 1$ から，$\theta = 90°$
$t = -\dfrac{\sqrt{3}}{2}$ で最小値 -6 をとり，
このとき，$\sin\theta = -\dfrac{\sqrt{3}}{2}$ から，$\theta = 240°$，$300°$
よって，

最大値：$\boldsymbol{1 + 4\sqrt{3}}$ ($\boldsymbol{\theta = 90°}$)，最小値：$\boldsymbol{-6}$ ($\boldsymbol{\theta = 240°}$，$\boldsymbol{300°}$)

【4】(1) $0^\circ \leqq \theta < 120^\circ$ より，

$$\begin{aligned} 0^\circ \leqq &\ \theta < 120^\circ \\ 0^\circ \leqq &\ 3\theta < 360^\circ \end{aligned}$$

ここで，$t = 3\theta$ とおくと，

$$0^\circ \leqq t < 360^\circ$$

$\sin 3\theta = 1$ より，

$$\begin{aligned} \sin t &= 1 \\ t &= 90^\circ \end{aligned}$$

$t = 90^\circ$ のとき，

$$\begin{aligned} 3\theta &= 90^\circ \\ \theta &= 30^\circ \end{aligned}$$

よって，$\boldsymbol{\theta = 30^\circ}$

(2) $0^\circ \leqq \theta < 315^\circ$ より，

$$\begin{aligned} 0^\circ \leqq &\ \theta < 315^\circ \\ 45^\circ \leqq &\ \theta + 45^\circ < 360^\circ \end{aligned}$$

ここで，$t = \theta + 45^\circ$ とおくと，

$$45^\circ \leqq t < 360^\circ$$

$\cos(\theta + 45^\circ) = \dfrac{1}{2}$ より，

$$\begin{aligned} \cos t &= \frac{1}{2} \\ t &= 60^\circ,\ 300^\circ \end{aligned}$$

$t = 60^\circ$ のとき，

$$\begin{aligned} \theta + 45^\circ &= 60^\circ \\ \theta &= 15^\circ \end{aligned}$$

$t = 300^\circ$ のとき，

$$\begin{aligned} \theta + 45^\circ &= 300^\circ \\ \theta &= 255^\circ \end{aligned}$$

よって，$\boldsymbol{\theta = 15^\circ,\ 255^\circ}$

(3) $15^\circ \leqq \theta \leqq 180^\circ$ より，

$$\begin{aligned} 15^\circ \leqq &\ \theta \leqq 180^\circ \\ 30^\circ \leqq &\ 2\theta \leqq 360^\circ \\ 0^\circ \leqq &\ 2\theta - 30^\circ \leqq 330^\circ \end{aligned}$$

ここで，$t = 2\theta - 30^\circ$ とおくと，

$$0^\circ \leqq t \leqq 330^\circ$$

$\sin(2\theta - 30^\circ) = -\dfrac{\sqrt{3}}{2}$ より，

$$\begin{aligned} \sin t &= -\frac{\sqrt{3}}{2} \\ t &= 240^\circ,\ 300^\circ \end{aligned}$$

$t = 240^\circ$ のとき，

$$\begin{aligned} 2\theta - 30^\circ &= 240^\circ \\ \theta &= 135^\circ \end{aligned}$$

$t = 300^\circ$ のとき，

$$\begin{aligned} 2\theta - 30^\circ &= 300^\circ \\ \theta &= 165^\circ \end{aligned}$$

よって，$\boldsymbol{\theta = 135^\circ,\ 165^\circ}$

(4) $20^\circ \leqq \theta \leqq 140^\circ$ より，

$$\begin{aligned} 20^\circ \leqq &\ \theta \leqq 140^\circ \\ 60^\circ \leqq &\ 3\theta \leqq 420^\circ \\ 0^\circ \leqq &\ 3\theta - 60^\circ \leqq 360^\circ \end{aligned}$$

ここで，$t = 3\theta - 60^\circ$ とおくと，

$$0^\circ \leqq t \leqq 360^\circ$$

$2\cos(3\theta - 60^\circ) + 5 = 4$ より，

$$\begin{aligned} 2\cos t + 5 &= 4 \\ \cos t &= -\frac{1}{2} \\ t &= 120^\circ,\ 240^\circ \end{aligned}$$

$t = 120^\circ$ のとき，

$$\begin{aligned} 3\theta - 60^\circ &= 120^\circ \\ \theta &= 60^\circ \end{aligned}$$

$t = 240^\circ$ のとき，

$$\begin{aligned} 3\theta - 60^\circ &= 240^\circ \\ \theta &= 100^\circ \end{aligned}$$

よって，$\boldsymbol{\theta = 60^\circ,\ 100^\circ}$

(5) $15^\circ < \theta < 105^\circ$ より，

$$
\begin{aligned}
15^\circ <& \quad \theta \quad < 105^\circ \\
30^\circ <& \quad 2\theta \quad < 210^\circ \\
90^\circ <& \ 2\theta + 60^\circ < 270^\circ
\end{aligned}
$$

ここで，$t = 2\theta + 60^\circ$ とおくと，

$$90^\circ < t < 270^\circ$$

$\tan(2\theta + 60^\circ) = -\dfrac{1}{\sqrt{3}}$ より，

$$
\begin{aligned}
\tan t &= -\frac{1}{\sqrt{3}} \\
t &= 150^\circ
\end{aligned}
$$

$t = 150^\circ$ のとき，

$$
\begin{aligned}
2\theta + 60^\circ &= 150^\circ \\
\theta &= 45^\circ
\end{aligned}
$$

よって，$\boldsymbol{\theta = 45^\circ}$

【5】(1) $0^\circ \leqq \theta < 180^\circ$ より

$$
\begin{aligned}
0^\circ \leqq& \ \ \theta \ < 180^\circ \\
0^\circ \leqq& \ 2\theta < 360^\circ
\end{aligned}
$$

ここで，$t = 2\theta$ とおくと

$$0^\circ \leqq t < 360^\circ$$

$\cos 2\theta > -\dfrac{1}{2}$ より

$$\cos t > -\frac{1}{2}$$

$$0^\circ \leqq t < 120^\circ,\ 240^\circ < t < 360^\circ$$

よって

$$\boldsymbol{0^\circ \leqq \theta < 60^\circ,\ 120^\circ < \theta < 180^\circ}$$

(2) $0^\circ \leqq \theta \leqq 315^\circ$ より

$$
\begin{aligned}
0^\circ \leqq& \quad \theta \quad \leqq 315^\circ \\
45^\circ \leqq& \ \theta + 45^\circ \leqq 360^\circ
\end{aligned}
$$

ここで，$t = \theta + 45^\circ$ とおくと

$$45^\circ \leqq t \leqq 360^\circ$$

$\sin(\theta + 45^\circ) \leqq \dfrac{\sqrt{3}}{2}$ より

$$\sin t \leqq \frac{\sqrt{3}}{2}$$

$$45^\circ \leqq t \leqq 60^\circ,\ 120^\circ \leqq t \leqq 360^\circ$$

よって

$$\boldsymbol{0^\circ \leqq \theta \leqq 15^\circ,\ 75^\circ \leqq \theta \leqq 315^\circ}$$

(3) $15^\circ \leqq \theta < 195^\circ$ より

$$\begin{aligned} 15^\circ \leqq &\quad \theta &< 195^\circ \\ 0^\circ \leqq &\ 2\theta - 30^\circ &< 360^\circ \end{aligned}$$

ここで，$t = 2\theta - 30^\circ$ とおくと

$0^\circ \leqq t < 360^\circ$

$\cos(2\theta - 30^\circ) < -\dfrac{1}{2}$ より

$\cos t < -\dfrac{1}{2}$

$$\begin{aligned} 120^\circ < &\quad t &< 240^\circ \\ 120^\circ < &\ 2\theta - 30^\circ &< 240^\circ \\ 150^\circ < &\quad 2\theta &< 270^\circ \\ 75^\circ < &\quad \theta &< 135^\circ \end{aligned}$$

よって

$\boldsymbol{75^\circ < \theta < 135^\circ}$

(4) $0^\circ \leqq \theta < 105^\circ$ より

$$\begin{aligned} 0^\circ \leqq &\quad \theta &< 105^\circ \\ 45^\circ \leqq &\ 3\theta + 45^\circ &< 360^\circ \end{aligned}$$

ここで，$t = 3\theta + 45^\circ$ とおくと

$45^\circ \leqq t < 360^\circ$

$2\sin(3\theta + 45^\circ) + \sqrt{3} \geqq 0$ より

$$\begin{aligned} 2\sin t &\geqq -\sqrt{3} \\ \sin t &\geqq -\frac{\sqrt{3}}{2} \end{aligned}$$

$45^\circ \leqq t \leqq 240^\circ$，$300^\circ \leqq t < 360^\circ$

なので

$$\begin{aligned} 45^\circ \leqq &\ 3\theta + 45^\circ &\leqq 240^\circ \\ 0^\circ \leqq &\quad 3\theta &\leqq 195^\circ \\ 0^\circ \leqq &\quad \theta &\leqq 65^\circ \end{aligned}$$

また

$$\begin{aligned} 300^\circ \leqq &\ 3\theta + 45^\circ &< 360^\circ \\ 255^\circ \leqq &\quad 3\theta &< 315^\circ \\ 85^\circ \leqq &\quad \theta &< 105^\circ \end{aligned}$$

よって

$\boldsymbol{0^\circ \leqq \theta \leqq 65^\circ,\ 85^\circ \leqq \theta < 105^\circ}$

(5) $105^\circ < \theta < 195^\circ$ より

$$\begin{aligned} 105^\circ < &\quad \theta &< 195^\circ \\ 90^\circ < &\ 2\theta - 120^\circ &< 270^\circ \end{aligned}$$

ここで，$t = 2\theta - 120^\circ$ とおくと

$90^\circ < t < 270^\circ$

$\tan(2\theta - 120^\circ) < -\sqrt{3}$ より

$\tan t < -\sqrt{3}$

$$\begin{aligned} 90^\circ < &\quad t &< 120^\circ \\ 90^\circ < &\ 2\theta - 120^\circ &< 120^\circ \\ 210^\circ < &\quad 2\theta &< 240^\circ \\ 105^\circ < &\quad \theta &< 120^\circ \end{aligned}$$

よって，$\boldsymbol{105^\circ < \theta < 120^\circ}$

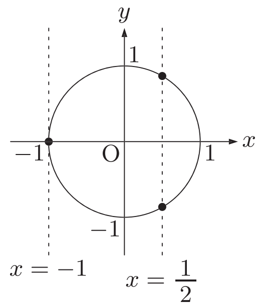
添削課題

【1】(1)

$$\sin\theta\tan\theta-1=\cos\theta$$
$$\sin\theta\times\frac{\sin\theta}{\cos\theta}-1=\cos\theta$$
$$\sin^2\theta-\cos\theta=\cos^2\theta$$
$$(1-\cos^2\theta)-\cos\theta=\cos^2\theta$$
$$2\cos^2\theta+\cos\theta-1=0$$
$$(2\cos\theta-1)(\cos\theta+1)=0$$
$$\cos\theta=\frac{1}{2},\ -1$$

よって，

$\boldsymbol{\theta=60^\circ,\ 180^\circ,\ 300^\circ}$

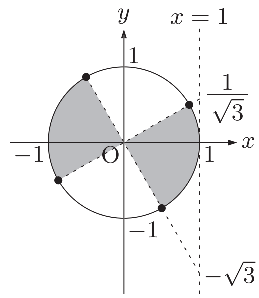


(2)

$$\sqrt{3}\tan^2\theta+2\tan\theta-\sqrt{3}\leqq 0$$
$$(\sqrt{3}\tan\theta-1)(\tan\theta+\sqrt{3})\leqq 0$$
$$-\sqrt{3}\leqq\tan\theta\leqq\frac{1}{\sqrt{3}}$$

よって，

$\boldsymbol{0^\circ\leqq\theta\leqq 30^\circ,\ 120^\circ\leqq\theta\leqq 210^\circ,}$
$\boldsymbol{300^\circ\leqq\theta<360^\circ}$



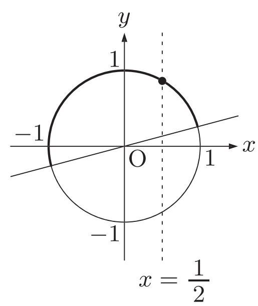
【2】(1) $0^\circ\leqq\theta\leqq 90^\circ$ より，$15^\circ\leqq 2\theta+15^\circ\leqq 195^\circ$

このもとで，

$$2\cos(2\theta+15^\circ)=1$$
$$\cos(2\theta+15^\circ)=\frac{1}{2}$$

を解くと，

$$2\theta+15^\circ=60^\circ$$
$$\therefore\ \boldsymbol{\theta=22.5^\circ}$$



(2) $0^\circ\leqq\theta\leqq 90^\circ$ より，$30^\circ\leqq 3\theta+30^\circ\leqq 300^\circ$

このもとで，

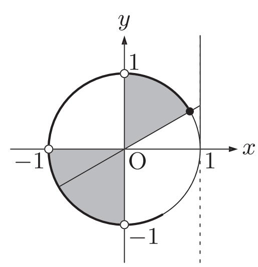
$$\tan(3\theta+30^\circ)>0$$

を解くと，

$30^\circ\leqq 3\theta+30^\circ<90^\circ$，$180^\circ<3\theta+30^\circ<270^\circ$

よって，

$\boldsymbol{0^\circ\leqq\theta<20^\circ,\ 50^\circ<\theta<80^\circ}$



【3】(1) $\sin\theta = 2\cos\theta$ を，$\sin^2\theta + \cos^2\theta = 1$ に代入して，

$$\begin{aligned} 4\cos^2\theta + \cos^2\theta &= 1 \\ 5\cos^2\theta &= 1 \\ \cos^2\theta &= \frac{1}{5} \end{aligned}$$

よって，

$$\sin\theta\cos\theta = 2\cos^2\theta = \boldsymbol{\frac{2}{5}}$$

＜別解＞

$\cos\theta = 0$ のとき，すなわち $\theta = 90^\circ$，270° のとき，$\sin\theta = \pm 1$ であるから，このとき与えられた等式

$$\sin\theta = 2\cos\theta \quad \cdots ①$$

は成り立たない．よって，$\cos\theta \fallingdotseq 0$

そこで，①の両辺を $\cos\theta$ で割ると，

$$\frac{\sin\theta}{\cos\theta} = \frac{2\cos\theta}{\cos\theta}$$

$$\tan\theta = 2$$

よって，

$$\cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{5}$$

であるから，

$$\sin\theta\cos\theta = \cos^2\theta\tan\theta = \frac{1}{5}\cdot 2 = \boldsymbol{\frac{2}{5}}$$

(2) $\cos\theta = 0$ のとき，$\sin\theta = 1$ だから，このとき与えられた等式

$$2\sin^2\theta + 2\sin\theta\cos\theta = 1 \quad \cdots ①$$

は成り立たない．よって，$\cos\theta \fallingdotseq 0$

そこで，①の両辺を $\cos^2\theta$ で割ると

$$2\frac{\sin^2\theta}{\cos^2\theta} + 2\frac{\sin\theta\cos\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\begin{aligned} 2\tan^2\theta + 2\tan\theta &= 1 + \tan^2\theta \\ \tan^2\theta + 2\tan\theta - 1 &= 0 \\ \tan\theta &= -1 \pm \sqrt{2} \end{aligned}$$

よって，

$$\begin{aligned} \tan(180^\circ - \theta) &= -\tan\theta \\ &= \boldsymbol{1 \pm \sqrt{2}} \end{aligned}$$

【4】 $y = 2 + a\sin x - \cos^2 x$

$= 2 + a\sin x - (1 - \sin^2 x)$

$= \sin^2 x + a\sin x + 1 \quad \cdots ①$

(1) $a = 1$ のとき,

$y = \sin^2 x + \sin x + 1$

$t = \sin x$ とおくと, $0° \leqq x \leqq 180°$ より, $0 \leqq t \leqq 1$ で,

$y = t^2 + t + 1$

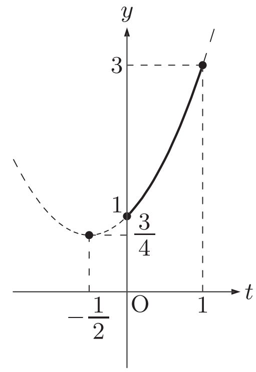
$= \left(t + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

グラフは右図のようになり, $t = 1$ のとき, すなわち,

$x = 90°$のとき, 最大値 3

$t = 0$ のとき, すなわち,

$x = 0°$, $180°$のとき, 最小値 1



(2) ①において, $t = \sin x$ とおくと, $0° \leqq x < 360°$ より $-1 \leqq t \leqq 1$ で,

$y = t^2 + at + 1$

$= \left(t + \dfrac{a}{2}\right)^2 + 1 - \dfrac{a^2}{4} \quad (a > 0)$

$= f(t)$

とおく.

(i) $-\dfrac{a}{2} \leqq -1$ のとき, すなわち, $a \geqq 2$ のとき, 最小値は,

$f(-1) = 2 - a$

これが -2 に等しいとき, $a = 4$ となり, 題意に適す.

(ii) $-\dfrac{a}{2} > -1$ のとき, すなわち, $0 < a < 2$ のとき, 最小値は,

$f\left(-\dfrac{a}{2}\right) = 1 - \dfrac{a^2}{4}$

これが -2 に等しいとき, $a^2 = 12$ より, $a = \pm 2\sqrt{3}$ となるが, $0 < a < 2$ に適さない.

(i)(ii) より, **$a = 4$**

２５章　三角比（５）ー正弦定理・余弦定理ー

問題

【1】(1) 正弦定理より，

$$\begin{aligned}\frac{a}{\sin A} &= \frac{c}{\sin C}\\ \frac{10}{\sin 45^\circ} &= \frac{c}{\sin 30^\circ}\\ \frac{10}{\frac{\sqrt{2}}{2}} &= \frac{c}{\frac{1}{2}}\\ \frac{\sqrt{2}}{2}c &= \frac{1}{2}\times 10\\ c &= \mathbf{5\sqrt{2}}\end{aligned}$$

(2) 正弦定理より，

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ \frac{6}{\sin 45^\circ} &= \frac{b}{\sin 60^\circ}\\ \frac{6}{\frac{\sqrt{2}}{2}} &= \frac{b}{\frac{\sqrt{3}}{2}}\\ \frac{\sqrt{2}}{2}b &= \frac{\sqrt{3}}{2}\times 6\\ b &= \mathbf{3\sqrt{6}}\end{aligned}$$

(3) $\angle\mathrm{B} = 120^\circ$，$\angle\mathrm{C} = 15^\circ$ より，

$$\begin{aligned}\angle\mathrm{A} &= 180^\circ - (\angle\mathrm{B} + \angle\mathrm{C})\\ &= 180^\circ - (120^\circ + 15^\circ)\\ &= 45^\circ\end{aligned}$$

なので，正弦定理より，

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ \frac{a}{\sin 45^\circ} &= \frac{9}{\sin 120^\circ}\\ \frac{a}{\frac{\sqrt{2}}{2}} &= \frac{9}{\frac{\sqrt{3}}{2}}\\ \frac{\sqrt{3}}{2}a &= \frac{\sqrt{2}}{2}\times 9\\ a &= \mathbf{3\sqrt{6}}\end{aligned}$$

(4) $\angle\mathrm{A} = 15^\circ$，$\angle\mathrm{B} = 135^\circ$ より，

$$\begin{aligned}\angle\mathrm{C} &= 180^\circ - (\angle\mathrm{A} + \angle\mathrm{B})\\ &= 180^\circ - (15^\circ + 135^\circ)\\ &= 30^\circ\end{aligned}$$

なので，正弦定理より，

$$\begin{aligned}\frac{b}{\sin B} &= \frac{c}{\sin C}\\ \frac{b}{\sin 135^\circ} &= \frac{6}{\sin 30^\circ}\\ \frac{b}{\frac{\sqrt{2}}{2}} &= \frac{6}{\frac{1}{2}}\\ \frac{1}{2}b &= \frac{\sqrt{2}}{2}\times 6\\ b &= \mathbf{6\sqrt{2}}\end{aligned}$$

【2】(1) 正弦定理より，

$$\begin{aligned}\frac{b}{\sin B} &= 2R\\ \frac{4}{\sin 30^\circ} &= 2R\\ \frac{4}{\frac{1}{2}} &= 2R\\ 2R &= 8\\ R &= \mathbf{4}\end{aligned}$$

(2) 正弦定理より，

$$\begin{aligned}\frac{c}{\sin C} &= 2R\\ \frac{\sqrt{2}}{\sin 45^\circ} &= 2R\\ \frac{\sqrt{2}}{\frac{\sqrt{2}}{2}} &= 2R\\ 2R &= 2\\ R &= \mathbf{1}\end{aligned}$$

(3) 正弦定理より，

$$\begin{aligned}\frac{a}{\sin A} &= 2R\\ \frac{6}{\sin 120^\circ} &= 2R\\ \frac{6}{\frac{\sqrt{3}}{2}} &= 2R\\ 2R &= 4\sqrt{3}\\ R &= \mathbf{2\sqrt{3}}\end{aligned}$$

(4) $\angle\mathrm{A} = 45^\circ$，$\angle\mathrm{C} = 105^\circ$ より，

$$\begin{aligned}\angle\mathrm{B} &= 180^\circ - (\angle\mathrm{A} + \angle\mathrm{C})\\ &= 180^\circ - (45^\circ + 105^\circ)\\ &= 30^\circ\end{aligned}$$

正弦定理より，

$$\begin{aligned}\frac{b}{\sin B} &= 2R\\ \frac{9}{\sin 30^\circ} &= 2R\\ \frac{9}{\frac{1}{2}} &= 2R\\ 2R &= 18\\ R &= \mathbf{9}\end{aligned}$$

(5) $\angle\mathrm{A} = 60^\circ$，$\angle\mathrm{B} = 75^\circ$ より，

$$\begin{aligned}\angle\mathrm{C} &= 180^\circ - (\angle\mathrm{A} + \angle\mathrm{B})\\ &= 180^\circ - (60^\circ + 75^\circ)\\ &= 45^\circ\end{aligned}$$

正弦定理より，

$$\begin{aligned}\frac{c}{\sin C} &= 2R\\ \frac{15}{\sin 45^\circ} &= 2R\\ \frac{15}{\frac{\sqrt{2}}{2}} &= 2R\\ 2R &= 15\sqrt{2}\\ R &= \mathbf{\frac{15\sqrt{2}}{2}}\end{aligned}$$

【3】 (1) 正弦定理より，

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ \frac{2}{\sin 30^\circ} &= \frac{2\sqrt{3}}{\sin B}\\ \frac{2}{\frac{1}{2}} &= \frac{2\sqrt{3}}{\sin B}\\ 2\sin B &= 2\sqrt{3} \times \frac{1}{2}\\ \sin B &= \frac{\sqrt{3}}{2}\\ \angle\mathrm{B} &= \mathbf{60^\circ，120^\circ}\end{aligned}$$

(2) 正弦定理より，

$$\begin{aligned}\frac{a}{\sin A} &= \frac{c}{\sin C}\\ \frac{2}{\sin A} &= \frac{\sqrt{2}}{\sin 30^\circ}\\ \frac{2}{\sin A} &= \frac{\sqrt{2}}{\frac{1}{2}}\\ \sqrt{2}\sin A &= 2 \times \frac{1}{2}\\ \sin A &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\\ \angle\mathrm{A} &= \mathbf{45^\circ，135^\circ}\end{aligned}$$

(3) 正弦定理より，

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{3\sqrt{2}}{\sin 45^\circ} = \frac{3}{\sin B}$$
$$\frac{3\sqrt{2}}{\frac{\sqrt{2}}{2}} = \frac{3}{\sin B}$$
$$3 \times \frac{\sqrt{2}}{2} = 3\sqrt{2}\sin B$$
$$\sin B = \frac{1}{2}$$
$$\angle \mathrm{B} = 30^\circ,\ 150^\circ$$

$\angle \mathrm{A} = 45^\circ$ より $\angle \mathrm{B} = 150^\circ$ は不適.
よって，$\angle \mathrm{B} = \mathbf{30^\circ}$

(4) 正弦定理より，

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{6}{\sin 60^\circ} = \frac{2\sqrt{6}}{\sin C}$$
$$\frac{6}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{6}}{\sin C}$$
$$2\sqrt{6} \times \frac{\sqrt{3}}{2} = 6\sin C$$
$$\sin C = \frac{\sqrt{2}}{2}$$
$$\angle \mathrm{C} = 45^\circ,\ 135^\circ$$

$\angle \mathrm{B} = 60^\circ$ より $\angle \mathrm{C} = 135^\circ$ は不適.
よって，$\angle \mathrm{C} = \mathbf{45^\circ}$

【4】(1) 余弦定理より，

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = (\sqrt{3})^2 + 5^2 - 2 \times \sqrt{3} \times 5 \times \cos 30^\circ$$
$$a^2 = (\sqrt{3})^2 + 5^2 - 2 \times \sqrt{3} \times 5 \times \frac{\sqrt{3}}{2}$$
$$a^2 = 3 + 25 - 15$$
$$a^2 = 13$$
$$a = \pm\sqrt{13}$$

$a > 0$ より，$a = \boldsymbol{\sqrt{13}}$

(2) 余弦定理より，

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$b^2 = (\sqrt{2})^2 + 2^2 - 2 \times \sqrt{2} \times 2 \times \cos 45^\circ$$
$$b^2 = (\sqrt{2})^2 + 2^2 - 2 \times \sqrt{2} \times 2 \times \frac{\sqrt{2}}{2}$$
$$b^2 = 2 + 4 - 4$$
$$b^2 = 2$$
$$b = \pm\sqrt{2}$$

$b > 0$ より，$b = \boldsymbol{\sqrt{2}}$

(3) 余弦定理より，

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 2^2 + 4^2 - 2 \times 2 \times 4 \times \cos 120^\circ$$
$$c^2 = 2^2 + 4^2 - 2 \times 2 \times 4 \times \left(-\frac{1}{2}\right)$$
$$c^2 = 4 + 16 + 8$$
$$c^2 = 28$$
$$c = \pm 2\sqrt{7}$$

$c > 0$ より，$c = \boldsymbol{2\sqrt{7}}$

(4) 余弦定理より，

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 4^2 + (\sqrt{2})^2 - 2 \times 4 \times \sqrt{2} \times \cos 135^\circ$$
$$a^2 = 4^2 + (\sqrt{2})^2 - 2 \times 4 \times \sqrt{2} \times \left(-\frac{\sqrt{2}}{2}\right)$$
$$a^2 = 16 + 2 + 8$$
$$a^2 = 26$$
$$a = \pm\sqrt{26}$$

$a > 0$ より，$a = \boldsymbol{\sqrt{26}}$

(5) 余弦定理より，

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
c^2 &= (2\sqrt{3})^2 + 4^2 - 2 \times 2\sqrt{3} \times 4 \times \cos 150^\circ \\
c^2 &= (2\sqrt{3})^2 + 4^2 - 2 \times 2\sqrt{3} \times 4 \times \left(-\frac{\sqrt{3}}{2}\right) \\
c^2 &= 12 + 16 + 24 \\
c^2 &= 52 \\
c &= \pm 2\sqrt{13}
\end{aligned}
$$

$c > 0$ より，$c = \mathbf{2\sqrt{13}}$

【5】(1) 余弦定理の変形より，

$$
\begin{aligned}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{3^2 + 8^2 - 7^2}{2 \cdot 3 \cdot 8} \\
&= \frac{9 + 64 - 49}{2 \cdot 3 \cdot 8} \\
&= \frac{24}{2 \cdot 3 \cdot 8} \\
&= \frac{1}{2} \\
\angle \mathrm{A} &= 60^\circ,\ 300^\circ
\end{aligned}
$$

三角形の内角の和は 180° なので，

$\angle \mathrm{A} = \mathbf{60^\circ}$

(2) 余弦定理の変形より

$$
\begin{aligned}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} \\
&= \frac{25 + 64 - 49}{2 \cdot 5 \cdot 8} \\
&= \frac{40}{2 \cdot 5 \cdot 8} \\
&= \frac{1}{2} \\
\angle \mathrm{A} &= 60^\circ,\ 300^\circ
\end{aligned}
$$

三角形の内角の和は 180° なので，

$\angle \mathrm{A} = \mathbf{60^\circ}$

(3) 余弦定理の変形より

$$
\begin{aligned}
\cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\
&= \frac{3^2 + (2\sqrt{2})^2 - (\sqrt{5})^2}{2 \cdot 3 \cdot 2\sqrt{2}} \\
&= \frac{9 + 8 - 5}{2 \cdot 3 \cdot 2\sqrt{2}} \\
&= \frac{12}{2 \cdot 3 \cdot 2\sqrt{2}} \\
&= \frac{\sqrt{2}}{2} \\
\angle \mathrm{B} &= 45^\circ,\ 315^\circ
\end{aligned}
$$

三角形の内角の和は 180° なので，

$\angle \mathrm{B} = \mathbf{45^\circ}$

(4) 余弦定理の変形より

$$
\begin{aligned}
\cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\
&= \frac{2^2 + (\sqrt{3}+1)^2 - (\sqrt{6})^2}{2 \cdot 2 \cdot (\sqrt{3}+1)} \\
&= \frac{4 + 4 + 2\sqrt{3} - 6}{2 \cdot 2 \cdot (\sqrt{3}+1)} \\
&= \frac{2 + 2\sqrt{3}}{2 \cdot 2 \cdot (\sqrt{3}+1)} \\
&= \frac{2(\sqrt{3}+1)}{2 \cdot 2 \cdot (\sqrt{3}+1)} \\
&= \frac{1}{2} \\
\angle \mathrm{C} &= 60^\circ,\ 300^\circ
\end{aligned}
$$

三角形の内角の和は 180° なので，

$\angle \mathrm{C} = \mathbf{60^\circ}$

【6】(1) 正弦定理より，

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$\frac{3}{\sin 60^\circ}=\frac{\sqrt{3}}{\sin B}$$
$$\frac{3}{\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{\sin B}$$
$$3\sin B=\frac{\sqrt{3}}{2}\times\sqrt{3}$$
$$\sin B=\frac{1}{2}$$
$$\angle \mathrm{B}=30^\circ,\ 150^\circ$$

$\angle \mathrm{A}=60^\circ$ より，$\angle \mathrm{B}<120^\circ$ なので，

$$\angle \mathrm{B}=\mathbf{30^\circ}$$

$\angle \mathrm{A}=60^\circ$，$\angle \mathrm{B}=30^\circ$ より，

$$\begin{aligned}\angle \mathrm{C}&=180^\circ-(\angle \mathrm{A}+\angle \mathrm{B})\\&=180^\circ-(60^\circ+30^\circ)\\&=\mathbf{90^\circ}\end{aligned}$$

余弦定理より，

$$\begin{aligned}&c^2\\=&a^2+b^2-2ab\cos C\\=&3^2+(\sqrt{3})^2-2\times 3\times\sqrt{3}\times\cos 90^\circ\\=&3^2+(\sqrt{3})^2-2\times 3\times\sqrt{3}\times 0\\=&12\\&c=\pm 2\sqrt{3}\end{aligned}$$

$c>0$ より，$c=\mathbf{2\sqrt{3}}$

よって，

$$\boldsymbol{c=2\sqrt{3}},\ \angle \mathrm{B}=\mathbf{30^\circ},\ \angle \mathrm{C}=\mathbf{90^\circ}$$

(2) 正弦定理より，

$$\frac{b}{\sin B}=\frac{c}{\sin C}$$
$$\frac{6}{\sin 30^\circ}=\frac{6\sqrt{3}}{\sin C}$$
$$\frac{6}{\frac{1}{2}}=\frac{6\sqrt{3}}{\sin C}$$
$$6\sin C=6\sqrt{3}\times\frac{1}{2}$$
$$\sin C=\frac{\sqrt{3}}{2}$$
$$\angle \mathrm{C}=\mathbf{60^\circ,\ 120^\circ}$$

$\angle \mathrm{C}=60^\circ$ のとき，

$$\begin{aligned}\angle \mathrm{A}&=180^\circ-(\angle \mathrm{B}+\angle \mathrm{C})\\&=180^\circ-(30^\circ+60^\circ)\\&=\mathbf{90^\circ}\end{aligned}$$

余弦定理より，

$$\begin{aligned}&a^2\\=&b^2+c^2-2bc\cos A\\=&6^2+(6\sqrt{3})^2-2\times 6\times 6\sqrt{3}\times\cos 90^\circ\\=&6^2+(6\sqrt{3})^2-2\times 6\times 6\sqrt{3}\times 0\\=&144\\&a=\pm 12\end{aligned}$$

$a>0$ より，$a=\mathbf{12}$

また，$\angle \mathrm{C}=120^\circ$ のとき，

$$\begin{aligned}\angle \mathrm{A}&=180^\circ-(\angle \mathrm{B}+\angle \mathrm{C})\\&=180^\circ-(30^\circ+120^\circ)\\&=30^\circ\end{aligned}$$

余弦定理より，

$$\begin{aligned}&a^2\\=&b^2+c^2-2bc\times\cos A\\=&6^2+(6\sqrt{3})^2-2\times 6\times 6\sqrt{3}\times\cos 30^\circ\\=&6^2+(6\sqrt{3})^2-2\times 6\times 6\sqrt{3}\times\frac{\sqrt{3}}{2}\\=&36+108-108\\=&36\\&a=\pm 6\end{aligned}$$

$a>0$ より，$a=\mathbf{6}$

よって，

$$\boldsymbol{a=12},\ \angle \mathrm{A}=\mathbf{90^\circ},\ \angle \mathrm{C}=\mathbf{60^\circ}$$

または，

$$\boldsymbol{a=6},\ \angle \mathrm{A}=\mathbf{30^\circ},\ \angle \mathrm{C}=\mathbf{120^\circ}$$

(3) 余弦定理より，

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
&= (2\sqrt{3})^2 + (3+\sqrt{3})^2 - 2 \times 2\sqrt{3} \times (3+\sqrt{3}) \times \cos 30^\circ \\
&= (2\sqrt{3})^2 + (3+\sqrt{3})^2 - 2 \times 2\sqrt{3} \times (3+\sqrt{3}) \times \frac{\sqrt{3}}{2} \\
&= 12 + 12 + 6\sqrt{3} - 6(3+\sqrt{3}) \\
&= 6 \\
c &= \pm\sqrt{6}
\end{aligned}
$$

$c > 0$ より，$c = \boldsymbol{\sqrt{6}}$

余弦定理の変形より，

$$
\begin{aligned}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{(3+\sqrt{3})^2 + (\sqrt{6})^2 - (2\sqrt{3})^2}{2 \cdot (3+\sqrt{3}) \cdot \sqrt{6}} \\
&= \frac{12 + 6\sqrt{3} + 6 - 12}{2 \cdot (3+\sqrt{3}) \cdot \sqrt{6}} \\
&= \frac{6 + 6\sqrt{3}}{2 \cdot (3+\sqrt{3}) \cdot \sqrt{6}} \\
&= \frac{6(1+\sqrt{3})}{2 \cdot \sqrt{3}(\sqrt{3}+1) \cdot \sqrt{6}} \\
&= \frac{1}{\sqrt{2}} \\
\angle \mathrm{A} &= 45^\circ,\ 315^\circ
\end{aligned}
$$

三角形の内角の和は 180° より，$\angle \mathrm{A} = \boldsymbol{45^\circ}$

$\angle \mathrm{A} = 45^\circ$，$\angle \mathrm{C} = 30^\circ$ より，

$$
\begin{aligned}
\angle \mathrm{B} &= 180^\circ - (\angle \mathrm{A} + \angle \mathrm{C}) \\
&= 180^\circ - (45^\circ + 30^\circ) \\
&= \boldsymbol{105^\circ}
\end{aligned}
$$

よって，$\boldsymbol{c = \sqrt{6}}$**，**$\angle \mathrm{A} = \boldsymbol{45^\circ}$**，**$\angle \mathrm{B} = \boldsymbol{105^\circ}$

(4) 余弦定理より，

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= (2\sqrt{3})^2 + (\sqrt{6}+\sqrt{2})^2 - 2 \times 2\sqrt{3} \times (\sqrt{6}+\sqrt{2}) \times \cos 45^\circ \\ &= (2\sqrt{3})^2 + (\sqrt{6}+\sqrt{2})^2 - 2 \times 2\sqrt{3} \times (\sqrt{6}+\sqrt{2}) \times \frac{\sqrt{2}}{2} \\ &= 12 + 8 + 4\sqrt{3} - 2\sqrt{6}(\sqrt{6}+\sqrt{2}) \\ &= 8 \\ a &= \pm 2\sqrt{2} \end{aligned}$$

$a > 0$ より，$a = \mathbf{2\sqrt{2}}$

余弦定理の変形より，

$$\begin{aligned} \cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\ &= \frac{(\sqrt{6}+\sqrt{2})^2 + (2\sqrt{2})^2 - (2\sqrt{3})^2}{2 \cdot (\sqrt{6}+\sqrt{2}) \cdot 2\sqrt{2}} \\ &= \frac{8 + 4\sqrt{3} + 8 - 12}{2 \cdot (\sqrt{6}+\sqrt{2}) \cdot 2\sqrt{2}} \\ &= \frac{4 + 4\sqrt{3}}{2 \cdot (\sqrt{6}+\sqrt{2}) \cdot 2\sqrt{2}} \\ &= \frac{4(1+\sqrt{3})}{2 \cdot \sqrt{2}(\sqrt{3}+1) \cdot 2\sqrt{2}} \\ &= \frac{1}{2} \\ \angle \mathrm{B} &= 60^\circ,\ 300^\circ \end{aligned}$$

三角形の内角の和は 180° より，$\angle \mathrm{B} = \mathbf{60^\circ}$

$\angle \mathrm{A} = 45^\circ$，$\angle \mathrm{B} = 60^\circ$ より，

$$\begin{aligned} \angle \mathrm{C} &= 180^\circ - (\angle \mathrm{A} + \angle \mathrm{B}) \\ &= 180^\circ - (45^\circ + 60^\circ) \\ &= \mathbf{75^\circ} \end{aligned}$$

よって，$\boldsymbol{a = 2\sqrt{2}}$**,** $\angle \mathrm{B} = \mathbf{60^\circ}$**,** $\angle \mathrm{C} = \mathbf{75^\circ}$

【7】(1) △BCD において，正弦定理より，

$$\frac{\sqrt{2}}{\sin 30^\circ} = \frac{\mathrm{BD}}{\sin 135^\circ}$$
$$\frac{\sqrt{2}}{\frac{1}{2}} = \frac{\mathrm{BD}}{\frac{\sqrt{2}}{2}}$$
$$\frac{1}{2}\mathrm{BD} = \frac{\sqrt{2}}{2} \times \sqrt{2}$$
$$\mathrm{BD} = \mathbf{2}$$

(2) △ABD において，余弦定理より，

$$\begin{aligned}\mathrm{AD}^2 &= 3^2 + 2^2 - 2 \times 3 \times 2 \times \cos 60^\circ \\ &= 3^2 + 2^2 - 2 \times 3 \times 2 \times \frac{1}{2} \\ &= 9 + 4 - 6 \\ &= 7\end{aligned}$$
$$\mathrm{AD} = \pm\sqrt{7}$$

$\mathrm{AD} > 0$ より，$\mathrm{AD} = \boldsymbol{\sqrt{7}}$

(3) △ABD において，余弦定理の変形より，

$$\begin{aligned}\cos A &= \frac{(\sqrt{7})^2 + 3^2 - 2^2}{2 \cdot \sqrt{7} \cdot 3} \\ &= \frac{7 + 9 - 4}{2 \cdot \sqrt{7} \cdot 3} \\ &= \frac{12}{2 \cdot \sqrt{7} \cdot 3} \qquad \therefore\ \cos A = \boldsymbol{\frac{2\sqrt{7}}{7}}\end{aligned}$$

【8】(1) △ABC において，余弦定理より，

$$\begin{aligned}\mathrm{AC}^2 &= 4^2 + 3^2 - 2 \times 4 \times 3 \times \cos 60^\circ \\ &= 4^2 + 3^2 - 2 \times 4 \times 3 \times \frac{1}{2} \\ &= 16 + 9 - 12 \\ &= 13\end{aligned}$$
$$\mathrm{AC} = \pm\sqrt{13}$$

$\mathrm{AC} > 0$ より，$\mathrm{AC} = \boldsymbol{\sqrt{13}}$

(2) 平行四辺形の内対角の和は 180° より，

$$\begin{aligned}\angle\mathrm{BCD} &= 180^\circ - \angle\mathrm{ABC} \\ &= 180^\circ - 60^\circ \\ &= 120^\circ\end{aligned}$$

△BCD において，余弦定理より，

$$\begin{aligned}\mathrm{BD}^2 &= 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 120^\circ \\ &= 3^2 + 4^2 - 2 \times 3 \times 4 \times \left(-\frac{1}{2}\right) \\ &= 9 + 16 + 12 \\ &= 37\end{aligned}$$
$$\mathrm{BD} = \pm\sqrt{37}$$

$\mathrm{BD} > 0$ より，$\mathrm{BD} = \boldsymbol{\sqrt{37}}$

【9】(1) △BCD において，余弦定理より，

$$\begin{aligned} \mathrm{BD}^2 &= 3^2+6^2-2\times3\times6\times\cos120^\circ \\ &= 3^2+6^2-2\times3\times6\times\left(-\frac{1}{2}\right) \\ &= 9+36+18 \\ &= 63 \\ \mathrm{BD} &= \pm3\sqrt{7} \end{aligned}$$

BD > 0 より，BD = $\mathbf{3\sqrt{7}}$

(2) 円に内接する四角形の対角の和は，180° より，

$$\begin{aligned} \angle\mathrm{BAD} &= 180^\circ-\angle\mathrm{BCD} \\ &= 180^\circ-120^\circ \\ &= 60^\circ \end{aligned}$$

△ABD において，正弦定理より，

$$\begin{aligned} \frac{3\sqrt{7}}{\sin60^\circ} &= \frac{\mathrm{AB}}{\sin45^\circ} \\ \frac{3\sqrt{7}}{\frac{\sqrt{3}}{2}} &= \frac{\mathrm{AB}}{\frac{\sqrt{2}}{2}} \\ \frac{\sqrt{3}}{2}\mathrm{AB} &= \frac{\sqrt{2}}{2}\cdot3\sqrt{7} \\ \mathrm{AB} &= \mathbf{\sqrt{42}} \end{aligned}$$

(3) 正弦定理より，

$$\begin{aligned} \frac{3\sqrt{7}}{\sin60^\circ} &= 2R \\ \frac{3\sqrt{7}}{\frac{\sqrt{3}}{2}} &= 2R \\ R &= \mathbf{\sqrt{21}} \end{aligned}$$

【10】(1)

$$\cos C = \frac{a^2+b^2-c^2}{2ab},\ \cos A = \frac{b^2+c^2-a^2}{2bc},\ \cos B = \frac{c^2+a^2-b^2}{2ca}$$

なので,

$$\begin{aligned}
\text{左辺} &= (a-b)(1+\cos C) \\
&= (a-b)\left(1+\frac{a^2+b^2-c^2}{2ab}\right) \\
&= (a-b)\left(\frac{2ab+a^2+b^2-c^2}{2ab}\right) \\
&= (a-b)\left\{\frac{(a+b)^2-c^2}{2ab}\right\} \\
&= (a-b)\left\{\frac{(a+b+c)(a+b-c)}{2ab}\right\} \\
&= \frac{(a-b)(a+b+c)(a+b-c)}{2ab}
\end{aligned}$$

また,

$$\begin{aligned}
\text{右辺} &= c(\cos B-\cos A) \\
&= c\left(\frac{c^2+a^2-b^2}{2ca}-\frac{b^2+c^2-a^2}{2bc}\right) \\
&= c\left\{\frac{b(c^2+a^2-b^2)-a(b^2+c^2-a^2)}{2abc}\right\} \\
&= \frac{b(a^2-b^2)+a(a^2-b^2)-c^2(a-b)}{2ab} \\
&= \frac{(a+b)(a^2-b^2)-c^2(a-b)}{2ab} \\
&= \frac{(a+b)^2(a-b)-c^2(a-b)}{2ab} \\
&= \frac{(a-b)(a+b+c)(a+b-c)}{2ab}
\end{aligned}$$

よって,左辺 $=$ 右辺 が成り立つので,

$$(a-b)(1+\cos C) = c(\cos B-\cos A)$$

(証明終)

(2)

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{a}{2R}}{\frac{b^2+c^2-a^2}{2bc}} = \frac{2abc}{2R(b^2+c^2-a^2)}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{b}{2R}}{\frac{c^2+a^2-b^2}{2ca}} = \frac{2abc}{2R(c^2+a^2-b^2)}$$

なので,

$$\begin{aligned} 左辺 &= (b^2+c^2-a^2)\tan A \\ &= (b^2+c^2-a^2)\cdot\frac{2abc}{2R(b^2+c^2-a^2)} \\ &= \frac{abc}{R} \end{aligned}$$

また,

$$\begin{aligned} 右辺 &= (c^2+a^2-b^2)\tan B \\ &= (c^2+a^2-b^2)\cdot\frac{2abc}{2R(c^2+a^2-b^2)} \\ &= \frac{abc}{R} \end{aligned}$$

よって，左辺 $=$ 右辺 が成り立つので,

$$(b^2+c^2-a^2)\tan A = (c^2+a^2-b^2)\tan B$$

（証明終）

(3)

$$\sin A = \frac{a}{2R},\ \sin B = \frac{b}{2R},\ \sin C = \frac{c}{2R},\ \cos A = \frac{b^2+c^2-a^2}{2bc}$$

なので,

$$\begin{aligned} 左辺 &= \sin^2 B + \sin^2 C - 2\sin B\sin C\cos A \\ &= \left(\frac{b}{2R}\right)^2 + \left(\frac{c}{2R}\right)^2 - 2\cdot\frac{b}{2R}\cdot\frac{c}{2R}\cdot\frac{b^2+c^2-a^2}{2bc} \\ &= \frac{b^2}{4R^2} + \frac{c^2}{4R^2} - \frac{b^2+c^2-a^2}{4R^2} \\ &= \frac{a^2}{4R^2} \\ &= \left(\frac{a}{2R}\right)^2 \\ &= \sin^2 A = 右辺 \end{aligned}$$

よって，左辺 $=$ 右辺 が成り立つので

$$\sin^2 B + \sin^2 C - 2\sin B\sin C\cos A = \sin^2 A$$

（証明終）

【11】(1) $\sin A = \dfrac{a}{2R}$, $\sin B = \dfrac{b}{2R}$ なので,

$$
\begin{aligned}
\sin A : \sin B &= b : a \\
\frac{a}{2R} : \frac{b}{2R} &= b : a \\
\frac{a}{2R} \times a &= \frac{b}{2R} \times b \\
\frac{a^2}{2R} &= \frac{b^2}{2R} \\
a^2 &= b^2 \\
a &= b
\end{aligned}
$$

これより, △ABC は,

$\boldsymbol{a = b}$ の二等辺三角形

(2) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ なので,

$$
\begin{aligned}
&a \cos B - b \cos A = c \\
&a \cdot \frac{c^2 + a^2 - b^2}{2ca} - b \cdot \frac{b^2 + c^2 - a^2}{2bc} = c \\
&\frac{c^2 + a^2 - b^2}{2c} - \frac{b^2 + c^2 - a^2}{2c} = c \\
&\frac{2a^2 - 2b^2}{2c} = c \\
&\frac{a^2 - b^2}{c} = c \\
&b^2 + c^2 = a^2
\end{aligned}
$$

これより, △ABC は,

∠A $= 90°$ の直角三角形

(3)

$$\sin B = \frac{b}{2R},\ \sin C = \frac{c}{2R},\ \cos B = \frac{c^2+a^2-b^2}{2ca},\ \cos C = \frac{a^2+b^2-c^2}{2ab}$$

なので，

$$\begin{aligned}
\sin B \cos B &= \sin C \cos C \\
\frac{b}{2R}\cdot\frac{c^2+a^2-b^2}{2ca} &= \frac{c}{2R}\cdot\frac{a^2+b^2-c^2}{2ab} \\
\frac{b(c^2+a^2-b^2)}{4Rac} &= \frac{c(a^2+b^2-c^2)}{4Rab} \\
4Rac^2(a^2+b^2-c^2) &= 4Rab^2(c^2+a^2-b^2) \\
c^2(a^2+b^2-c^2) &= b^2(c^2+a^2-b^2) \\
a^2c^2-c^4 &= a^2b^2-b^4 \\
(a^2c^2-a^2b^2)-(c^4-b^4) &= 0 \\
a^2(c^2-b^2)-(c^2+b^2)(c^2-b^2) &= 0 \\
(c^2-b^2)(a^2-c^2-b^2) &= 0 \\
(c+b)(c-b)(a^2-c^2-b^2) &= 0
\end{aligned}$$

これより，

$$\begin{cases} b+c=0 & \cdots ① \\ c-b=0 & \cdots ② \\ a^2-c^2-b^2=0 & \cdots ③ \end{cases}$$

である．①は，不適．

②は，

$$c-b=0 \quad \therefore\ c=b\ \cdots ②'$$

③は，

$$a^2-c^2-b^2=0 \quad \therefore\ a^2=c^2+b^2\ \cdots ③'$$

よって，△ABC は，**$\boldsymbol{b=c}$ の二等辺三角形もしくは ∠A $\boldsymbol{= 90^\circ}$ の直角三角形**

【12】(1) △ABC において，余弦定理より，

$$\begin{aligned}\mathrm{BC}^2 &= 3^2 + 2^2 - 2\times3\times2\times\cos60^\circ\\ &= 3^2+2^2-2\times3\times2\times\frac{1}{2}\\ &= 9+4-6\\ &= 7\end{aligned}$$

$$\mathrm{BC} = \pm\sqrt{7}$$

$\mathrm{BC} > 0$ より，$\mathrm{BC} = \sqrt{7}$

よって，

$$\mathrm{BM} = \frac{1}{2}\mathrm{BC} = \boldsymbol{\frac{\sqrt{7}}{2}}$$

(2) △ABC において，余弦定理の変形より，

$$\begin{aligned}\cos B &= \frac{3^2+(\sqrt{7})^2-2^2}{2\cdot3\cdot\sqrt{7}}\\ &= \frac{9+7-4}{2\cdot3\cdot\sqrt{7}}\\ &= \frac{12}{2\cdot3\cdot\sqrt{7}}\\ &= \boldsymbol{\frac{2}{7}\sqrt{7}}\end{aligned}$$

(3) △ABM において，余弦定理より，

$$\begin{aligned}\mathrm{AM}^2 &= \left(\frac{\sqrt{7}}{2}\right)^2 + 3^2 - 2\times\frac{\sqrt{7}}{2}\times3\times\cos B\\ &= \left(\frac{\sqrt{7}}{2}\right)^2 + 3^2 - 2\times\frac{\sqrt{7}}{2}\times3\times\frac{2}{7}\sqrt{7}\\ &= \frac{7}{4}+9-6\\ &= \frac{19}{4}\end{aligned}$$

$$\mathrm{AM} = \pm\frac{\sqrt{19}}{2}$$

$\mathrm{AM} > 0$ より，$\mathrm{AM} = \boldsymbol{\frac{\sqrt{19}}{2}}$

【13】(1) △ABC と △ADC において，

$$\begin{aligned}5^2+4^2-2\times5\times4\times\cos B &= 3^2+4^2-2\times3\times4\times\cos D\\ 5^2+4^2-2\times5\times4\times\cos B &= 3^2+4^2-2\times3\times4\times\cos(180^\circ-B)\\ 5^2+4^2-2\times5\times4\times\cos B &= 3^2+4^2-2\times3\times4\times(-\cos B)\\ 25+16-40\cos B &= 9+16+24\cos B\\ 64\cos B &= 16\\ \cos B &= \boldsymbol{\frac{1}{4}}\end{aligned}$$

(2) $\sin^2 B+\cos^2 B=1$ より，

$$\sin^2 B+\left(\frac{1}{4}\right)^2=1$$
$$\sin^2 B=\frac{15}{16}$$
$$\sin B=\pm\frac{\sqrt{15}}{4}$$

$0^\circ<B<180^\circ$ より，$\sin B>0$ だから，

$$\sin B=\frac{\sqrt{15}}{4}$$

△ABC において，余弦定理より，

$$\begin{aligned}\mathrm{AC}^2&=5^2+4^2-2\times5\times4\times\cos B\\&=5^2+4^2-2\times5\times4\times\frac{1}{4}\\&=25+16-10\\&=31\end{aligned}$$
$$\mathrm{AC}=\pm\sqrt{31}$$

$\mathrm{AC}>0$ より，$\mathrm{AC}=\sqrt{31}$ 正弦定理より，

$$\frac{\mathrm{AC}}{\sin B}=2R$$
$$\frac{\sqrt{31}}{\frac{\sqrt{15}}{4}}=2R$$
$$R=\mathbf{\frac{2\sqrt{465}}{15}}$$

【14】(1) $a:b:c=3:5:6$ より，

$$a=3k,\ b=5k,\ c=6k\quad(k>0)$$

とする．正弦定理より，

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$
$$\frac{3k}{\sin A}=\frac{5k}{\sin B}=\frac{6k}{\sin C}=2R$$

だから，

$$\sin A=\frac{3k}{2R},\ \sin B=\frac{5k}{2R},\ \sin C=\frac{6k}{2R}$$

よって，

$$\begin{aligned}\sin A:\sin B:\sin C&=\frac{3k}{2R}:\frac{5k}{2R}:\frac{6k}{2R}\\&=\mathbf{3:5:6}\end{aligned}$$

(2) $a : b : c = 3 : 5 : 6$ より，

$$a = 3k,\ b = 5k,\ c = 6k \quad (k > 0)$$

とする．余弦定理の変形より，

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{(5k)^2 + (6k)^2 - (3k)^2}{2 \cdot 5k \cdot 6k} \\ &= \frac{25k^2 + 36k^2 - 9k^2}{2 \cdot 5k \cdot 6k} \\ &= \frac{52k^2}{60k^2} = \boldsymbol{\frac{13}{15}}\end{aligned}$$

$$\begin{aligned}\cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\ &= \frac{(6k)^2 + (3k)^2 - (5k)^2}{2 \cdot 6k \cdot 3k} \\ &= \frac{36k^2 + 9k^2 - 25k^2}{2 \cdot 6k \cdot 3k} \\ &= \frac{20k^2}{36k^2} = \boldsymbol{\frac{5}{9}}\end{aligned}$$

$$\begin{aligned}\cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{(3k)^2 + (5k)^2 - (6k)^2}{2 \cdot 3k \cdot 5k} \\ &= \frac{9k^2 + 25k^2 - 36k^2}{2 \cdot 3k \cdot 5k} \\ &= \frac{-2k^2}{30k^2} = -\boldsymbol{\frac{1}{15}}\end{aligned}$$

【15】(1) 条件式の連比を，$\dfrac{a+b}{7}=\dfrac{b+c}{11}=\dfrac{c+a}{8}=k$ とおくと，

$$\begin{cases} a+b=7k & \cdots ① \\ b+c=11k & \cdots ② \\ c+a=8k & \cdots ③ \end{cases}$$

$(①+②+③)\div 2$ より，

$$a+b+c=13k\cdots ④$$

④ − ①，④ − ②，④ − ③ より，

$$a=2k,\ b=5k,\ c=6k\cdots ⑤$$

よって，正弦定理より，

$$\sin A:\sin B:\sin C=\frac{a}{2R}:\frac{b}{2R}:\frac{c}{2R}=a:b:c$$

だから，

$$\sin A:\sin B:\sin C=2k:5k:6k=\mathbf{2:5:6}$$

(2) 余弦定理より，

$$\cos A=\frac{(5k)^2+(6k)^2-(2k)^2}{2\cdot 5k\cdot 6k}=\frac{57k^2}{60k^2}=\frac{19}{20}$$

より，

$$\sin A=\sqrt{1-\left(\frac{19}{20}\right)^2}=\sqrt{1-\frac{361}{400}}=\boldsymbol{\frac{\sqrt{39}}{20}}$$

(1) より，

$$\sin B=\frac{\sqrt{39}}{20}\times\frac{5}{2}=\boldsymbol{\frac{\sqrt{39}}{8}}$$

$$\sin C=\frac{\sqrt{39}}{20}\times\frac{6}{2}=\boldsymbol{\frac{3\sqrt{39}}{20}}$$

添削課題

【1】(1) 余弦定理より，

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 2^2 + (1+\sqrt{3})^2 - 2\times 2\times(1+\sqrt{3})\times\cos 60^\circ = 6$$

$a > 0$ より，$a = \sqrt{6}$

正弦定理より，$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ だから，

$$\frac{\sqrt{6}}{\sin 60^\circ} = \frac{2}{\sin B} = 2R$$

よって，$R = \dfrac{\sqrt{6}}{2\sin 60^\circ} = \sqrt{2}$，$\sin B = \dfrac{1}{\sqrt{2}}$ より，$B = 45^\circ$，135°

ここで，$A = 60^\circ$ より，$B < 120^\circ$ だから，$B = 45^\circ$

したがって，

$$C = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$$

以上より，

$\boldsymbol{B = 45^\circ}$**，**$\boldsymbol{C = 75^\circ}$**，**$\boldsymbol{a = \sqrt{6}}$**，**$\boldsymbol{R = \sqrt{2}}$

(2) $A = 180^\circ - (45^\circ + 105^\circ) = 30^\circ$

正弦定理より，$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = 2R$ だから，

$$\frac{4}{\sin 30^\circ} = \frac{b}{\sin 45^\circ} = 2R$$

よって，$8 = \sqrt{2}b$ より，$b = 4\sqrt{2}$

さらに，$R = \dfrac{2}{\sin 30^\circ} = 4$

余弦定理より，

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$4^2 = (4\sqrt{2})^2 + c^2 - 2\times 4\sqrt{2}\times c\times\cos 30^\circ$$
$$16 = 32 + c^2 - 4\sqrt{6}c$$

よって，$c^2 - 4\sqrt{6}c + 16 = 0$ より，$c = 2\sqrt{6}\pm 2\sqrt{2}$

ここで，C が最大角より，c は最大辺となるので，$c = 2\sqrt{6} + 2\sqrt{2}$

以上より，

$\boldsymbol{A = 30^\circ}$**，**$\boldsymbol{b = 4\sqrt{2}}$**，**$\boldsymbol{c = 2\sqrt{6} + 2\sqrt{2}}$**，**$\boldsymbol{R = 4}$

【2】(1) $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ なので,

$$
\begin{aligned}
a\cos B - b\cos A &= c \\
a\cdot\frac{c^2+a^2-b^2}{2ca} - b\cdot\frac{b^2+c^2-a^2}{2bc} &= c \\
\frac{c^2+a^2-b^2}{2c} - \frac{b^2+c^2-a^2}{2c} &= c \\
\frac{2a^2-2b^2}{2c} &= c \\
\frac{a^2-b^2}{c} &= c \\
a^2-b^2 &= c^2 \\
b^2+c^2 &= a^2
\end{aligned}
$$

これより, △ABC は, **∠A = 90°の直角三角形**

(2) $\sin B = \dfrac{b}{2R}$, $\sin C = \dfrac{c}{2R}$, $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

なので,

$$
\begin{aligned}
\sin B\cos B &= \sin C\cos C \\
\frac{b}{2R}\cdot\frac{c^2+a^2-b^2}{2ca} &= \frac{c}{2R}\cdot\frac{a^2+b^2-c^2}{2ab} \\
\frac{b(c^2+a^2-b^2)}{4Rac} &= \frac{c(a^2+b^2-c^2)}{4Rab} \\
4Rac^2(a^2+b^2-c^2) &= 4Rab^2(c^2+a^2-b^2) \\
c^2(a^2+b^2-c^2) &= b^2(c^2+a^2-b^2)
\end{aligned}
$$

$$
\begin{aligned}
a^2c^2-c^4 &= a^2b^2-b^4 \\
(a^2c^2-a^2b^2)-(c^4-b^4) &= 0 \\
a^2(c^2-b^2)-(c^2+b^2)(c^2-b^2) &= 0 \\
(c^2-b^2)(a^2-c^2-b^2) &= 0 \\
(c+b)(c-b)(a^2-c^2-b^2) &= 0
\end{aligned}
$$

これより,

$$
\begin{cases}
b+c=0 & \cdots① \\
c-b=0 & \cdots② \\
a^2-c^2-b^2=0 & \cdots③
\end{cases}
$$

である. ①は, 不適.

②は,

$c-b=0$ $\quad\therefore\ c=b\ \cdots②'$

③は,

$a^2-c^2-b^2=0$ $\quad\therefore\ a^2=c^2+b^2\ \cdots③'$

よって, △ABC は, **$b=c$ の二等辺三角形もしくは ∠A = 90°の直角三角形**

【3】(1) 正弦定理より，

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

よって，

$$\begin{aligned} a : b : c &= 2R\sin A : 2R\sin B : 2R\sin C \\ &= \sin A : \sin B : \sin C \\ &= \mathbf{7 : 8 : 13} \end{aligned}$$

(2) $a = 7k,\ b = 8k,\ c = 13k$ とおくと，
最大辺 c の対角 C が最大角だから，余弦定理より，

$$\begin{aligned} \cos C &= \frac{(7k)^2 + (8k)^2 - (13k)^2}{2 \times 7k \times 8k} \\ &= \frac{49k^2 + 64k^2 - 169k^2}{112k^2} \\ &= \frac{-56k^2}{112k^2} \\ &= -\frac{1}{2} \end{aligned}$$

よって，$\boldsymbol{C = 120^\circ}$

【4】(1) $\mathrm{BM} = \mathrm{CM} = a$，$\mathrm{AC} = b$，$\mathrm{AB} = c$，$\mathrm{AM} = l$，$\angle\mathrm{AMB} = \theta$ とおく．
$\triangle\mathrm{ABM}$ で余弦定理より，

$$\cos\theta = \frac{l^2 + a^2 - c^2}{2al} \quad \cdots ①$$

$\triangle\mathrm{ACM}$ において，$\angle\mathrm{AMC} = 180^\circ - \theta$ だから，余弦定理より，

$$\cos(180^\circ - \theta) = \frac{l^2 + a^2 - b^2}{2al} \quad \cdots ②$$

①，②において，$\cos(180^\circ - \theta) = -\cos\theta$ より，

$$\begin{aligned} -\frac{l^2 + a^2 - b^2}{2al} &= \frac{l^2 + a^2 - c^2}{2al} \\ -l^2 - a^2 + b^2 &= l^2 + a^2 - c^2 \\ c^2 + b^2 &= 2(l^2 + a^2) \end{aligned}$$

したがって，

$$\mathrm{AB}^2 + \mathrm{AC}^2 = 2(\mathrm{AM}^2 + \mathrm{BM}^2)$$ （証明終）

が成り立つ．

(2) (1) より，

$$\begin{aligned} 7^2 + 5^2 &= 2(\mathrm{AM}^2 + 3^2) \\ \mathrm{AM}^2 &= 28 \\ \therefore\ \mathrm{AM} &= 2\sqrt{7} \end{aligned}$$

よって，

$$\mathrm{AG} = \frac{2}{3}\mathrm{AM} = \boldsymbol{\frac{4\sqrt{7}}{3}}$$

問題

【1】(1)

$$\begin{aligned} S &= \frac{1}{2}bc\sin A \\ &= \frac{1}{2}\times 6\times 4\times \sin 60^\circ \\ &= \frac{1}{2}\times 6\times 4\times \frac{\sqrt{3}}{2} \\ &= \mathbf{6\sqrt{3}} \end{aligned}$$

(2)

$$\begin{aligned} S &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}\times 3\times 4\times \sin 45^\circ \\ &= \frac{1}{2}\times 3\times 4\times \frac{\sqrt{2}}{2} \\ &= \mathbf{3\sqrt{2}} \end{aligned}$$

(3)

$$\begin{aligned} S &= \frac{1}{2}ac\sin B \\ &= \frac{1}{2}\times 2\sqrt{3}\times 5\times \sin 120^\circ \\ &= \frac{1}{2}\times 2\sqrt{3}\times 5\times \frac{\sqrt{3}}{2} \\ &= \mathbf{\frac{15}{2}} \end{aligned}$$

(4)

$$\begin{aligned} S &= \frac{1}{2}bc\sin A \\ &= \frac{1}{2}\times \sqrt{2}\times 4\times \sin 135^\circ \\ &= \frac{1}{2}\times \sqrt{2}\times 4\times \frac{\sqrt{2}}{2} \\ &= \mathbf{2} \end{aligned}$$

【2】(1)

$$\begin{aligned} \cos A &= \frac{8^2+9^2-5^2}{2\times 8\times 9} \\ &= \frac{120}{144} \\ &= \frac{5}{6} \end{aligned}$$

よって,

$$\begin{aligned} \sin^2 A &= 1-\left(\frac{5}{6}\right)^2 \\ &= \frac{11}{36} \\ \sin A &= \pm\frac{\sqrt{11}}{6} \end{aligned}$$

$\sin A > 0$ より,

$$\sin A = \frac{\sqrt{11}}{6}$$

△ABC の面積は,

$$\begin{aligned} S &= \frac{1}{2}bc\sin A \\ &= \frac{1}{2}\times 8\times 9\times \frac{\sqrt{11}}{6} \\ &= \mathbf{6\sqrt{11}} \end{aligned}$$

(2)

$$\begin{aligned} \cos A &= \frac{3^2+4^2-2^2}{2\times 3\times 4} \\ &= \frac{21}{24} \\ &= \frac{7}{8} \end{aligned}$$

よって,

$$\begin{aligned} \sin^2 A &= 1-\left(\frac{7}{8}\right)^2 \\ &= \frac{15}{64} \\ \sin A &= \pm\frac{\sqrt{15}}{8} \end{aligned}$$

$\sin A > 0$ より,

$$\sin A = \frac{\sqrt{15}}{8}$$

△ABC の面積は,

$$\begin{aligned} S &= \frac{1}{2}bc\sin A \\ &= \frac{1}{2}\times 3\times 4\times \frac{\sqrt{15}}{8} \\ &= \mathbf{\frac{3\sqrt{15}}{4}} \end{aligned}$$

(3)

$$\cos A = \frac{2^2 + 1^2 - \left(\sqrt{2}\right)^2}{2 \times 2 \times 1} = \frac{3}{4}$$

よって,

$$\sin^2 A = 1 - \left(\frac{3}{4}\right)^2$$
$$= \frac{7}{16}$$
$$\sin A = \pm\frac{\sqrt{7}}{4}$$

$\sin A > 0$ より,

$$\sin A = \frac{\sqrt{7}}{4}$$

△ABC の面積は,

$$S = \frac{1}{2}bc\sin A$$
$$= \frac{1}{2} \times 2 \times 1 \times \frac{\sqrt{7}}{4}$$
$$= \boldsymbol{\frac{\sqrt{7}}{4}}$$

(4) $\cos A$

$$= \frac{\left(2\sqrt{3}\right)^2 + \left(3+\sqrt{3}\right)^2 - \left(\sqrt{6}\right)^2}{2 \times 2\sqrt{3} \times \left(3+\sqrt{3}\right)}$$
$$= \frac{12 + 12 + 6\sqrt{3} - 6}{4\sqrt{3}\left(3+\sqrt{3}\right)}$$
$$= \frac{18 + 6\sqrt{3}}{4\sqrt{3}(3+\sqrt{3})}$$
$$= \frac{6\left(\sqrt{3}+3\right)}{4\sqrt{3}\left(\sqrt{3}+3\right)}$$
$$= \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2}$$

よって,

$$\sin^2 A = 1 - \cos^2 A$$
$$= 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{4}$$
$$\sin A = \pm\frac{1}{2}$$

$\sin A > 0$ より,

$$\sin A = \frac{1}{2}$$

△ABC の面積は,

$$S = \frac{1}{2}bc \times \sin A$$
$$= \frac{1}{2} \times 2\sqrt{3} \times \left(3+\sqrt{3}\right) \times \frac{1}{2}$$
$$= \boldsymbol{\frac{3\sqrt{3}+3}{2}}$$

【3】(1) 余弦定理より，

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 3^2 + 8^2 - 2 \times 3 \times 8 \times \cos 60^\circ$$
$$c^2 = 3^2 + 8^2 - 2 \times 3 \times 8 \times \frac{1}{2}$$
$$c^2 = 9 + 64 - 24$$
$$c^2 = 49$$
$$c = \pm 7$$

$c > 0$ より，$c = 7$

△ABC の面積は，

$$\begin{aligned} S &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2} \times 3 \times 8 \times \sin 60^\circ \\ &= \frac{1}{2} \times 3 \times 8 \times \frac{\sqrt{3}}{2} \\ &= 6\sqrt{3} \end{aligned}$$

また，

$$\begin{aligned} \frac{1}{2}r(a+b+c) &= S \\ \frac{1}{2}r(3+7+8) &= 6\sqrt{3} \\ 9r &= 6\sqrt{3} \\ r &= \boldsymbol{\frac{2}{3}\sqrt{3}} \end{aligned}$$

正弦定理より，

$$2R = \frac{c}{\sin C}$$
$$2R = \frac{7}{\sin 60^\circ}$$
$$R = \frac{7}{2\sin 60^\circ}$$
$$R = \frac{7}{2 \times \frac{\sqrt{3}}{2}}$$
$$R = \frac{7}{\sqrt{3}}$$
$$R = \boldsymbol{\frac{7\sqrt{3}}{3}}$$

(2) 余弦定理より，

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 120^\circ$$
$$c^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \left(-\frac{1}{2}\right)$$
$$c^2 = 64 + 49 + 56$$
$$c^2 = 169$$
$$c = \pm 13$$

$c > 0$ より，$c = 13$

△ABC の面積は，

$$\begin{aligned} S &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2} \times 8 \times 7 \times \sin 120^\circ \\ &= \frac{1}{2} \times 8 \times 7 \times \frac{\sqrt{3}}{2} \\ &= 14\sqrt{3} \end{aligned}$$

また，

$$\begin{aligned} S &= \frac{1}{2}r(a+b+c) \\ 14\sqrt{3} &= \frac{1}{2}r(8+7+13) \\ 14\sqrt{3} &= 14r \\ r &= \boldsymbol{\sqrt{3}} \end{aligned}$$

正弦定理より，

$$2R = \frac{c}{\sin C}$$
$$R = \frac{13}{2\sin 120^\circ}$$
$$R = \frac{13}{2 \times \frac{\sqrt{3}}{2}}$$
$$R = \frac{13}{\sqrt{3}}$$
$$R = \boldsymbol{\frac{13\sqrt{3}}{3}}$$

(3)

$$\begin{aligned}\cos A &= \frac{b^2+c^2-a^2}{2bc}\\ &= \frac{5^2+6^2-4^2}{2\times5\times6}\\ &= \frac{45}{60}\\ &= \frac{3}{4}\end{aligned}$$

よって,

$$\begin{aligned}\sin^2 A &= 1-\left(\frac{3}{4}\right)^2\\ &= 1-\frac{9}{16}\\ &= \frac{7}{16}\\ \sin A &= \pm\frac{\sqrt{7}}{4}\end{aligned}$$

$\sin A > 0$ より, $\sin A = \frac{\sqrt{7}}{4}$

△ABC の面積は,

$$\begin{aligned}S &= \frac{1}{2}bc\sin A\\ &= \frac{1}{2}\times5\times6\times\frac{\sqrt{7}}{4}\\ &= \frac{15\sqrt{7}}{4}\end{aligned}$$

また,

$$\begin{aligned}S &= \frac{1}{2}r(a+b+c)\\ \frac{15\sqrt{7}}{4} &= \frac{1}{2}r(4+5+6)\\ \frac{15}{2}r &= \frac{15\sqrt{7}}{4}\\ r &= \boldsymbol{\frac{\sqrt{7}}{2}}\end{aligned}$$

正弦定理より,

$$\begin{aligned}2R &= \frac{a}{\sin A}\\ R &= \frac{a}{2\sin A}\\ R &= \frac{4}{2\times\frac{\sqrt{7}}{4}}\\ R &= \frac{8}{\sqrt{7}}\\ R &= \boldsymbol{\frac{8\sqrt{7}}{7}}\end{aligned}$$

(4)

$$\begin{aligned}\cos A &= \frac{b^2+c^2-a^2}{2bc}\\ &= \frac{5^2+7^2-3^2}{2\times5\times7}\\ &= \frac{65}{70} = \frac{13}{14}\end{aligned}$$

よって,

$$\begin{aligned}\sin^2 A &= 1-\left(\frac{13}{14}\right)^2\\ &= \frac{14^2-13^2}{14^2}\\ &= \frac{(14+13)(14-13)}{196}\\ &= \frac{27}{196}\\ \sin A &= \pm\frac{3\sqrt{3}}{14}\end{aligned}$$

$\sin A > 0$ より, $\sin A = \frac{3\sqrt{3}}{14}$

△ABC の面積は,

$$\begin{aligned}S &= \frac{1}{2}bc\sin A\\ &= \frac{1}{2}\times5\times7\times\frac{3\sqrt{3}}{14}\\ &= \frac{15\sqrt{3}}{4}\end{aligned}$$

また,

$$\begin{aligned}S &= \frac{1}{2}r(a+b+c)\\ \frac{15\sqrt{3}}{4} &= \frac{1}{2}r(3+5+7)\\ \frac{15}{2}r &= \frac{15\sqrt{3}}{4}\\ r &= \boldsymbol{\frac{\sqrt{3}}{2}}\end{aligned}$$

正弦定理より,

$$\begin{aligned}2R &= \frac{a}{\sin A}\\ R &= \frac{a}{2\sin A}\\ R &= \frac{3}{2\times\frac{3\sqrt{3}}{14}}\\ R &= \frac{7}{\sqrt{3}}\\ R &= \boldsymbol{\frac{7}{3}\sqrt{3}}\end{aligned}$$

【4】(1) 頂点 A から △BCD に垂線 AH を下ろす.

△BCD において，正弦定理より，

$$\begin{aligned} 2\mathrm{BH} &= \frac{10}{\sin 60^\circ} \\ \mathrm{BH} &= \frac{10}{2\sin 60^\circ} \\ &= \frac{10}{2\times\frac{\sqrt{3}}{2}} \\ &= \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3} \end{aligned}$$

△ABH において，三平方の定理より，

$$\begin{aligned} \mathrm{AH} &= \sqrt{10^2 - \left(\frac{10\sqrt{3}}{3}\right)^2} \\ &= \sqrt{100 - \frac{100}{3}} \\ &= \frac{10\sqrt{6}}{3} \end{aligned}$$

一方，

$$\begin{aligned} \triangle\mathrm{BCD} &= \frac{1}{2}\times 10\times 10\times\sin 60^\circ \\ &= \frac{1}{2}\times 10\times 10\times\frac{\sqrt{3}}{2} \\ &= 25\sqrt{3} \end{aligned}$$

したがって，

$$\begin{aligned} V &= \frac{1}{3}\times 25\sqrt{3}\times\frac{10\sqrt{6}}{3} \\ &= \boldsymbol{\frac{250\sqrt{2}}{3}} \end{aligned}$$

(2) 頂点 A から △BCD に垂線 AH を下ろす.

△BCD において，正弦定理より，

$$\begin{aligned} 2\mathrm{BH} &= \frac{6}{\sin 60^\circ} \\ \mathrm{BH} &= \frac{6}{2\sin 60^\circ} \\ &= \frac{6}{2\times\frac{\sqrt{3}}{2}} \\ &= \frac{6}{\sqrt{3}} = 2\sqrt{3} \end{aligned}$$

△ABH において，三平方の定理より，

$$\begin{aligned} \mathrm{AH} &= \sqrt{4^2 - (2\sqrt{3})^2} \\ &= \sqrt{16-12} \\ &= 2 \end{aligned}$$

一方，

$$\begin{aligned} \triangle\mathrm{BCD} &= \frac{1}{2}\times 6\times 6\times\sin 60^\circ \\ &= \frac{1}{2}\times 6\times 6\times\frac{\sqrt{3}}{2} \\ &= 9\sqrt{3} \end{aligned}$$

したがって，

$$\begin{aligned} V &= \frac{1}{3}\times 9\sqrt{3}\times 2 \\ &= \boldsymbol{6\sqrt{3}} \end{aligned}$$

(3) 頂点 A から △BCD に垂線 AH を下ろす.

△BCD において，正弦定理より，

$$\begin{aligned} 2\mathrm{BH} &= \frac{6}{\sin 60^\circ} \\ \mathrm{BH} &= \frac{6}{2\sin 60^\circ} \\ &= \frac{6}{2 \times \frac{\sqrt{3}}{2}} \\ &= \frac{6}{\sqrt{3}} = 2\sqrt{3} \end{aligned}$$

△ABH において，三平方の定理より，

$$\begin{aligned} \mathrm{AH} &= \sqrt{8^2 - (2\sqrt{3})^2} \\ &= \sqrt{64 - 12} \\ &= \sqrt{52} = 2\sqrt{13} \end{aligned}$$

一方，

$$\begin{aligned} \triangle\mathrm{BCD} &= \frac{1}{2} \times 6 \times 6 \times \sin 60^\circ \\ &= \frac{1}{2} \times 6 \times 6 \times \frac{\sqrt{3}}{2} \\ &= 9\sqrt{3} \end{aligned}$$

したがって，

$$\begin{aligned} V &= \frac{1}{3} \times 9\sqrt{3} \times 2\sqrt{13} \\ &= \mathbf{6\sqrt{39}} \end{aligned}$$

(4) 頂点 A から △BCD に垂線 AH を下ろす.

すると，H は △BCD の外接円の中心になる.

ここで，余弦定理より，

$$6^2 = 4^2 + (2\sqrt{7})^2 - 2 \cdot 4 \cdot 2\sqrt{7} \cdot \cos C$$

$$36 = 16 + 28 - 16\sqrt{7}\cos C$$

$$\begin{aligned} 16\sqrt{7}\cos C &= 8 \\ \cos C &= \frac{1}{2\sqrt{7}} \\ &= \frac{\sqrt{7}}{14} \end{aligned}$$

$0^\circ < C < 180^\circ$ より，$\sin C > 0$ なので，

$$\begin{aligned} \sin C &= \sqrt{1 - \left(\frac{\sqrt{7}}{14}\right)^2} \\ &= \sqrt{\frac{189}{196}} \\ &= \frac{3\sqrt{21}}{14} \end{aligned}$$

したがって，正弦定理より，

$$\begin{aligned} \frac{6}{\frac{3\sqrt{21}}{14}} &= 2R \\ R &= \frac{2\sqrt{21}}{3} \end{aligned}$$

△ACH において，三平方の定理より，

$$\begin{aligned} \mathrm{AH} &= \sqrt{6^2 - \left(\frac{2\sqrt{21}}{3}\right)^2} \\ &= \frac{4\sqrt{15}}{3} \end{aligned}$$

また，△BCD の面積を S とすると，

$$\begin{aligned} S &= \frac{1}{2} \times \mathrm{BC} \times \mathrm{CD} \times \sin C \\ &= \frac{1}{2} \times 4 \times 2\sqrt{7} \times \frac{3\sqrt{21}}{14} \\ &= 6\sqrt{3} \end{aligned}$$

よって，体積 V は，

$$\begin{aligned} V &= \frac{1}{3} \times 6\sqrt{3} \times \frac{4\sqrt{15}}{3} \\ &= \mathbf{8\sqrt{5}} \end{aligned}$$

【5】(1) 頂点 A から $\triangle$BCD に垂線 AH を下ろす.

$$\begin{aligned} 2\mathrm{BH} &= \frac{12}{\sin 60^\circ} \\ \mathrm{BH} &= \frac{12}{2\sin 60^\circ} \\ &= \frac{12}{2\times\dfrac{\sqrt{3}}{2}} \\ &= \frac{12}{\sqrt{3}} = 4\sqrt{3} \end{aligned}$$

$\triangle$ABH において，三平方の定理より，

$$\begin{aligned} \mathrm{AH} &= \sqrt{12^2-(4\sqrt{3})^2} \\ &= \sqrt{144-48} \\ &= \sqrt{96} \\ &= 4\sqrt{6} \end{aligned}$$

一方，

$$\begin{aligned} \triangle\mathrm{BCD} &= \frac{1}{2}\times 12\times 12\times\sin 60^\circ \\ &= \frac{1}{2}\times 12\times 12\times\frac{\sqrt{3}}{2} \\ &= 36\sqrt{3} \end{aligned}$$

したがって，三角すいの体積は

$$\frac{1}{3}\times 36\sqrt{3}\times 4\sqrt{6} = 144\sqrt{2}$$

合同な 4 つの正三角形で囲まれた立体図形だから，

$$\triangle\mathrm{ABC} = \triangle\mathrm{BCD} = \triangle\mathrm{ADB} = \triangle\mathrm{BCD} = 36\sqrt{3}$$

よって，

$$\begin{aligned} \frac{1}{3}\times r\times(36\sqrt{3}\times 4) &= 144\sqrt{2} \\ r\times(12\sqrt{3}\times 4) &= 144\sqrt{2} \\ r &= \boldsymbol{\sqrt{6}} \end{aligned}$$

球の体積は，

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(\sqrt{6})^3 \\ &= \boldsymbol{8\sqrt{6}\pi} \end{aligned}$$

球の表面積は，

$$\begin{aligned} S &= 4\pi r^2 \\ &= 4\pi(\sqrt{6})^2 \\ &= \boldsymbol{24\pi} \end{aligned}$$

(2)
$$\begin{aligned}\mathrm{BC} = \mathrm{CD} = \mathrm{DB} &= \sqrt{6^2+6^2}\\ &= 6\sqrt{2}\end{aligned}$$

三角すいの底面を △ABC とし，高さ AD とすると，三角すいの体積は，

$$\frac{1}{3}\times\left(\frac{1}{2}\times 6\times 6\right)\times 6 = 36$$

ここで，

$$\triangle\mathrm{ABC} = \triangle\mathrm{ABD} = \triangle\mathrm{ACD} = \frac{1}{2}\times 6\times 6 = 18$$
$$\begin{aligned}\triangle\mathrm{BCD} &= \frac{1}{2}\times 6\sqrt{2}\times 6\sqrt{2}\times\sin 60^\circ\\ &= 18\sqrt{3}\end{aligned}$$

三角すいは，4 つの三角形で囲まれた立体図形だから，

$$\begin{aligned}\frac{1}{3}\times r\times(18\times 3+18\sqrt{3}) &= 36\\ r\times 18\times(3+\sqrt{3}) &= 108\\ r &= \mathbf{3-\sqrt{3}}\end{aligned}$$

球の体積は，

$$\begin{aligned}V &= \frac{4}{3}\pi r^3\\ &= \frac{4}{3}\pi(3-\sqrt{3})^3\\ &= \frac{4}{3}\pi(54-30\sqrt{3})\\ &= \boldsymbol{(72-40\sqrt{3})\pi}\end{aligned}$$

球の表面積は，

$$\begin{aligned}S &= 4\pi r^2\\ &= 4\pi(3-\sqrt{3})^2\\ &= \boldsymbol{(48-24\sqrt{3})\pi}\end{aligned}$$

(3) 頂点 A から △BCD に垂線 AH を下ろす.

$$2\mathrm{BH} = \frac{8}{\sin 60^\circ}$$
$$\mathrm{BH} = \frac{8}{2\sin 60^\circ} = \frac{8}{2 \times \dfrac{\sqrt{3}}{2}} = \frac{8\sqrt{3}}{3}$$

△ABH において，三平方の定理より，

$$\mathrm{AH} = \sqrt{5^2 - \left(\frac{8\sqrt{3}}{3}\right)^2} = \sqrt{25 - \frac{192}{9}} = \frac{\sqrt{33}}{3}$$

一方，

$$\triangle\mathrm{BCD} = \frac{1}{2} \times 8 \times 8 \times \sin 60^\circ = \frac{1}{2} \times 8 \times 8 \times \frac{\sqrt{3}}{2} = 16\sqrt{3}$$

したがって，三角すいの体積は，

$$\frac{1}{3} \times 16\sqrt{3} \times \frac{\sqrt{33}}{3} = \frac{16\sqrt{11}}{3}$$

また，

$$\triangle\mathrm{ABC} = \triangle\mathrm{ACD} = \triangle\mathrm{ABD} = 12$$

より，

$$\frac{1}{3} \times r \times (12 \times 3 + 16\sqrt{3}) = \frac{16\sqrt{11}}{3}$$
$$r \times (36 + 16\sqrt{3}) = 16\sqrt{11}$$
$$r = \frac{16\sqrt{11}}{36 + 16\sqrt{3}} = \boldsymbol{\frac{36\sqrt{11} - 16\sqrt{33}}{33}}$$

球の体積は，

$$V = \frac{4}{3}\pi r^3 = \boldsymbol{\frac{4}{3}\left(\frac{36\sqrt{11} - 16\sqrt{33}}{33}\right)^3 \pi}$$

球の表面積は，

$$S = 4\pi r^2 = \boldsymbol{4\left(\frac{36\sqrt{11} - 16\sqrt{33}}{33}\right)^2 \pi}$$

(4) 三角すいの体積は,

$$\frac{1}{3}\times\left(2\times2\times\frac{1}{2}\right)\times4=\frac{8}{3}$$

また,

$$\begin{aligned}\mathrm{CD}&=\sqrt{2^2+2^2}\\&=2\sqrt{2}\\\mathrm{AC}&=\mathrm{AD}\\&=\sqrt{4^2+2^2}\\&=2\sqrt{5}\end{aligned}$$

△ACD において, A から CD に垂線 AH をおろすと,

$$\begin{aligned}\mathrm{AH}&=\sqrt{(2\sqrt{5})^2-(\sqrt{2})^2}\\&=3\sqrt{2}\end{aligned}$$

ここで, 4 つの面の面積は,

$$\begin{aligned}\triangle\mathrm{ABC}&=2\times4\times\frac{1}{2}=4\\\triangle\mathrm{ABD}&=4\times2\times\frac{1}{2}=4\\\triangle\mathrm{BCD}&=2\times2\times\frac{1}{2}=2\\\triangle\mathrm{ACD}&=2\sqrt{2}\times3\sqrt{2}\times\frac{1}{2}=6\end{aligned}$$

三角すいの体積は,

$$\begin{aligned}\frac{1}{3}r(4+4+2+6)&=\frac{8}{3}\\\frac{16}{3}r&=\frac{8}{3}\\r&=\boldsymbol{\frac{1}{2}}\end{aligned}$$

球の体積は,

$$\begin{aligned}V&=\frac{4}{3}\pi r^3\\&=\frac{4}{3}\pi\times\left(\frac{1}{2}\right)^3\\&=\boldsymbol{\frac{1}{6}\pi}\end{aligned}$$

球の表面積は,

$$\begin{aligned}S&=4\pi r^2\\&=4\pi\times\left(\frac{1}{2}\right)^2\\&=\boldsymbol{\pi}\end{aligned}$$

【6】(1) △ABC において，余弦定理より，

$$\cos A = \frac{8^2 + 7^2 - 13^2}{2 \times 8 \times 7}$$
$$= \frac{-56}{112}$$
$$= -\frac{1}{2}$$
$$\angle \mathrm{A} = 120^\circ,\ 240^\circ$$

三角形の内角の和は 180° なので，

$$\angle \mathrm{A} = \mathbf{120^\circ}$$

(2) $\mathrm{AD} = x$ とおく．

$$S = \triangle \mathrm{ABD} + \triangle \mathrm{ACD}$$
$$= \frac{1}{2} \cdot 8 \cdot x \cdot \sin 60^\circ + \frac{1}{2} \cdot 7 \cdot x \cdot \sin 60^\circ$$
$$= 2\sqrt{3}x + \frac{7\sqrt{3}}{4}x$$
$$= \frac{15\sqrt{3}}{4}x$$

また，

$$S = \frac{1}{2} \cdot 8 \cdot 7 \cdot \sin 120^\circ$$
$$= 14\sqrt{3}$$

よって，

$$\frac{15\sqrt{3}}{4}x = 14\sqrt{3}$$
$$x = 14\sqrt{3} \times \frac{4}{15\sqrt{3}}$$
$$= \mathbf{\frac{56}{15}}$$

【7】(1)

$$\triangle \mathrm{ABC} = \frac{1}{2} \times \mathrm{AB} \times \mathrm{AC} \times \sin \angle \mathrm{A}$$
$$= \frac{1}{2} \times \sqrt{3} \times 2 \times \sin 60^\circ$$
$$= \mathbf{\frac{3}{2}}$$

(2) $\mathrm{AD} = x$ とおく．

$$S = \triangle \mathrm{ABD} + \triangle \mathrm{ADC}$$
$$= \frac{1}{2} \cdot \sqrt{3} \cdot x \cdot \sin 30^\circ + \frac{1}{2} \cdot 2 \cdot x \cdot \sin 30^\circ$$
$$= \frac{\sqrt{3}}{4}x + \frac{1}{2}x$$
$$= \frac{\sqrt{3} + 2}{4}x$$

また，(1) より，

$$\frac{\sqrt{3} + 2}{4}x = \frac{3}{2}$$
$$x = \frac{3}{2} \times \frac{4}{\sqrt{3} + 2}$$
$$= \mathbf{12 - 6\sqrt{3}}$$

【8】(1)

$$
\begin{aligned}
\mathrm{AF} &= \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{6}\right)^2} \\
&= 3 \\
\mathrm{FC} &= \sqrt{1^2 + \left(\sqrt{6}\right)^2} \\
&= \sqrt{7} \\
\mathrm{AC} &= \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \\
&= 2
\end{aligned}
$$

より，$\triangle\mathrm{AFC}$ において，余弦定理より，

$$
\begin{aligned}
\cos\alpha &= \frac{3^2 + \left(\sqrt{7}\right)^2 - 2^2}{2 \times 3 \times \sqrt{7}} \\
&= \frac{12}{6\sqrt{7}} \\
&= \frac{2\sqrt{7}}{7}
\end{aligned}
$$

よって，

$$
\begin{aligned}
\sin^2\alpha &= 1 - \cos^2\alpha \\
&= 1 - \left(\frac{2\sqrt{7}}{7}\right)^2 \\
&= \frac{21}{49} \\
\sin\alpha &= \pm\frac{\sqrt{21}}{7}
\end{aligned}
$$

$\sin\alpha > 0$ より，

$$
\sin\alpha = \boldsymbol{\frac{\sqrt{21}}{7}}
$$

(2)

$$
\begin{aligned}
S &= \frac{1}{2} \times \mathrm{AF} \times \mathrm{FC} \times \sin\alpha \\
&= \frac{1}{2} \times 3 \times \sqrt{7} \times \frac{\sqrt{21}}{7} \\
&= \boldsymbol{\frac{3\sqrt{3}}{2}}
\end{aligned}
$$

(3) 三角すい $\mathrm{B-ACF}$ の体積を V とすると，

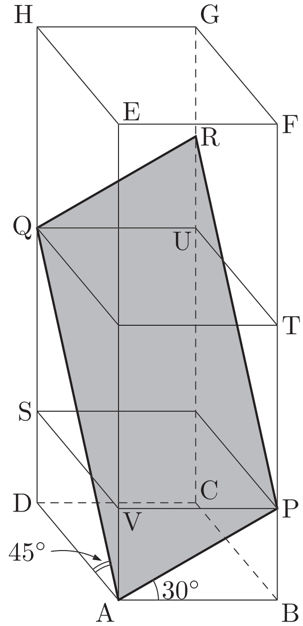
$$
\begin{aligned}
V &= \triangle\mathrm{ACF} \times \ell \times \frac{1}{3} \\
&= \triangle\mathrm{ABC} \times \mathrm{BF} \times \frac{1}{3}
\end{aligned}
$$

より，

$$
\begin{aligned}
S \times \ell \times \frac{1}{3} &= \left(\frac{1}{2} \times \sqrt{3} \times 1\right) \times \sqrt{6} \times \frac{1}{3} \\
\frac{3\sqrt{3}}{2} \times \ell \times \frac{1}{3} &= \frac{\sqrt{2}}{2} \\
\therefore\ \ell &= \boldsymbol{\frac{\sqrt{6}}{3}}
\end{aligned}
$$

【9】(1) まず，

$$\begin{aligned} PB &= 3\tan 30^\circ \\ &= \sqrt{3} \\ QD &= 3\tan 60^\circ \\ &= 3\sqrt{3} \end{aligned}$$



ここで，

$$\begin{aligned} QS &= QD - DS \\ &= QD - PB \\ &= 3\sqrt{3} - \sqrt{3} \\ &= 2\sqrt{3} \end{aligned}$$

△PSV において，三平方の定理より，

$$\begin{aligned} PS &= \sqrt{3^2 + 3^2} \\ &= 3\sqrt{2} \end{aligned}$$

△QSP において，三平方の定理より，

$$\begin{aligned} PQ &= \sqrt{(2\sqrt{3})^2 + (3\sqrt{2})^2} \\ &= \sqrt{12 + 18} \\ &= \boldsymbol{\sqrt{30}} \end{aligned}$$

(2) △APB において，

$$\begin{aligned} AP &= \sqrt{3^2 + (\sqrt{3})^2} \\ &= \sqrt{9 + 3} \\ &= \sqrt{12} = 2\sqrt{3} \end{aligned}$$

△AQD において，

$$\begin{aligned} AQ &= \sqrt{3^2 + (3\sqrt{3})^2} \\ &= \sqrt{9 + 27} \\ &= \sqrt{36} = 6 \end{aligned}$$

△APQ において，余弦定理の変形より，

$$\begin{aligned} \cos\alpha &= \frac{6^2 + (2\sqrt{3})^2 - (\sqrt{30})^2}{2 \cdot 6 \cdot 2\sqrt{3}} \\ &= \boldsymbol{\frac{\sqrt{3}}{4}} \end{aligned}$$

(3) (2) より，

$$\begin{aligned} \sin\alpha &= \sqrt{1 - \left(\frac{\sqrt{3}}{4}\right)^2} \\ &= \sqrt{1 - \frac{3}{16}} \\ &= \sqrt{\frac{13}{16}} \\ &= \frac{\sqrt{13}}{4} \end{aligned}$$

平行四辺形 APRQ の面積を S とすると，

$$\begin{aligned} S &= 2\triangle APQ \\ &= 2 \times \left(\frac{1}{2} \times 2\sqrt{3} \times 6 \times \frac{\sqrt{13}}{4}\right) \\ &= \boldsymbol{3\sqrt{39}} \end{aligned}$$

添削課題

【1】(1)

$$S = \frac{1}{2} \times 3 \times 4 \times \sin 60^\circ$$
$$= \mathbf{3\sqrt{3}}$$

また，

$$b^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos 60^\circ$$
$$= 13$$
$$\therefore\ b = \sqrt{13}$$

よって，内接円の半径を r とすると，

$$\frac{1}{2}r\left(3 + 4 + \sqrt{13}\right) = 3\sqrt{3}$$
$$\left(7 + \sqrt{13}\right)r = 6\sqrt{3}$$
$$r = \frac{6\sqrt{3}}{7 + \sqrt{13}}$$
$$= \frac{6\sqrt{3}\left(7 - \sqrt{13}\right)}{\left(7 + \sqrt{13}\right)\left(7 - \sqrt{13}\right)}$$
$$= \frac{6\sqrt{3}\left(7 - \sqrt{13}\right)}{36}$$
$$= \mathbf{\frac{7\sqrt{3} - \sqrt{39}}{6}}$$

(2) 正弦定理より，

$$\frac{b}{\sin 45^\circ} = \frac{4}{\sin 60^\circ}$$
$$b = \frac{4\sqrt{6}}{3}$$

よって，

$$S = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2} \cdot 4 \cdot \frac{4\sqrt{6}}{3} \cdot \sin 75^\circ$$
$$= \frac{8\sqrt{6}}{3} \cdot \frac{\sqrt{6} + \sqrt{2}}{4}$$
$$= \mathbf{4 + \frac{4\sqrt{3}}{3}}$$

【2】(1) AD $= x$ とおくと，$\triangle$ABD $+$ $\triangle$ACD $=$ $\triangle$ABC より，

$$\frac{1}{2}\cdot 15\cdot x\cdot \sin 30^\circ + \frac{1}{2}\cdot 13\cdot x\cdot \sin 30^\circ = \frac{1}{2}\cdot 15\cdot 13\cdot \sin 60^\circ$$

$$7x = \frac{195\sqrt{3}}{4}$$

$$\therefore\ x = \boldsymbol{\frac{195\sqrt{3}}{28}}$$

(2) BD : CD $=$ AB : AC $= 15 : 13$ で，BC $= 14$ より，

$$\text{BD} = \frac{15}{15+13}\times 14 = \frac{15}{2}$$

AD $= x$ とおくと，$\triangle$ABD で余弦定理から，

$$\cos B = \frac{15^2 + \left(\frac{15}{2}\right)^2 - x^2}{2\cdot 15\cdot \frac{15}{2}} = \frac{1125 - 4x^2}{900}$$

また，$\triangle$ABC で余弦定理から，

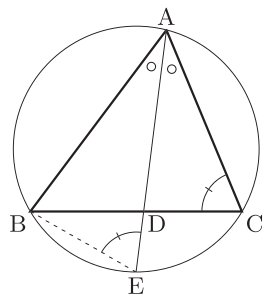
$$\cos B = \frac{15^2 + 14^2 - 13^2}{2\cdot 15\cdot 14} = \frac{3}{5}$$

よって，

$$\frac{1125 - 4x^2}{900} = \frac{3}{5} \qquad \therefore\ x = \boldsymbol{\frac{3\sqrt{65}}{2}}$$

＜別解＞

右の図のように，AD の延長と $\triangle$ABC の外接円との交点を E とし，AD $= x$，DE $= y$ とおく．



BD $= \dfrac{15}{2}$，CD $= \dfrac{13}{2}$ だから，方べきの定理より，

$$xy = \frac{15}{2}\cdot\frac{13}{2}\ \cdots ①$$

また，$\triangle$ABE ∽ $\triangle$ADC より，

$$\text{AE} : \text{AC} = \text{AB} : \text{AD}$$

$$\text{AD}\times\text{AE} = \text{AB}\times\text{AC}$$

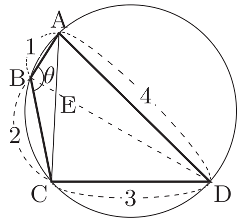
$$x(x+y) = 15\cdot 13\ \cdots ②$$

①を②に代入して，

$$x^2 = \frac{3}{4}\cdot 15\cdot 13$$

$$x = \boldsymbol{\frac{3}{2}\sqrt{65}}$$

【3】(1) $\angle ABC = \theta$ とおくと，$\triangle ABC$ で余弦定理から，



$$AC^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos\theta$$
$$= 5 - 4\cos\theta \quad \cdots ①$$

$\angle ADC = 180^\circ - \theta$ だから，$\triangle ADC$ で余弦定理より，

$$AC^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos(180^\circ - \theta)$$
$$= 25 + 24\cos\theta \quad \cdots ②$$

①，②より，

$$5 - 4\cos\theta = 25 + 24\cos\theta$$
$$\cos\theta = -\frac{5}{7} \quad \cdots ③$$

③を①に代入して，

$$AC^2 = 5 + \frac{20}{7} = \frac{55}{7} \qquad \therefore\ AC = \frac{\sqrt{55}}{\sqrt{7}} = \boldsymbol{\frac{\sqrt{385}}{7}}$$

(2) ③より，

$$\sin\theta = \sqrt{1 - \left(-\frac{5}{7}\right)^2} = \frac{2\sqrt{6}}{7}$$

円 O の半径，すなわち，$\triangle ABC$ の外接円の半径を R とすると，正弦定理から，

$$2R = \frac{AC}{\sin\theta} = \frac{\frac{\sqrt{385}}{7}}{\frac{2\sqrt{6}}{7}} = \frac{\sqrt{385}}{2\sqrt{6}} = \frac{\sqrt{2310}}{12} \qquad \therefore\ R = \boldsymbol{\frac{\sqrt{2310}}{24}}$$

(3)

$$\triangle ABC = \frac{1}{2} \cdot 1 \cdot 2 \cdot \sin\theta = \frac{2\sqrt{6}}{7}$$
$$\triangle ADC = \frac{1}{2} \cdot 3 \cdot 4 \cdot \sin(180^\circ - \theta) = 6\sin\theta = \frac{12\sqrt{6}}{7}$$

したがって，四角形 ABCD の面積 S は，

$$S = \triangle ABC + \triangle ADC = \boldsymbol{2\sqrt{6}}$$

(4)

$BE : ED = \triangle ABC : \triangle ADC = 1 : 6$ より，$BE = \frac{1}{7}BD \quad \cdots ④$

だから，(1) と同様にして，BD の長さを求めればよい．

$\angle BCD = \alpha$ とおくと，

$$BD^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \cos\alpha$$
$$= 1^2 + 4^2 - 2 \cdot 1 \cdot 4 \cdot \cos(180^\circ - \alpha)$$
$$13 - 12\cos\alpha = 17 + 8\cos\alpha$$
$$\cos\alpha = -\frac{1}{5}$$

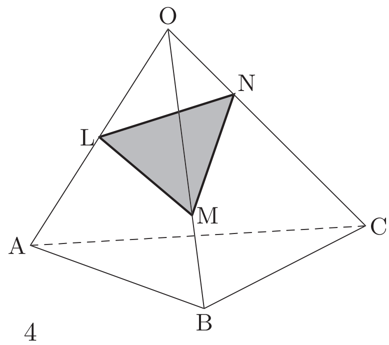
よって，

$$BD^2 = 13 + \frac{12}{5} = \frac{77}{5} \qquad \therefore\ BD = \frac{\sqrt{77}}{\sqrt{5}}$$

④より，

$$BE = \frac{\sqrt{77}}{7\sqrt{5}} = \frac{\sqrt{11}}{\sqrt{35}} = \boldsymbol{\frac{\sqrt{385}}{35}}$$

【4】(1) △OLM，△OMN，△ONL において，余弦定理より，



$$\mathrm{LM}^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 60^\circ = 13$$
$$\mathrm{MN}^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \times \cos 60^\circ = 12$$
$$\mathrm{NL}^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos 60^\circ = 7$$

よって，

$$\mathrm{LM} = \sqrt{13}, \quad \mathrm{MN} = 2\sqrt{3}, \quad \mathrm{NL} = \sqrt{7}$$

△LMN において，余弦定理より，

$$\cos \angle \mathrm{MLN} = \frac{(\sqrt{13})^2 + (\sqrt{7})^2 - (2\sqrt{3})^2}{2 \times \sqrt{13} \times \sqrt{7}} = \frac{4}{\sqrt{91}}$$

よって，

$$\sin \angle \mathrm{MLN} = \sqrt{1^2 - \left(\frac{4}{\sqrt{91}}\right)^2} = \frac{5\sqrt{3}}{\sqrt{91}}$$

したがって，

$$\triangle \mathrm{LMN} = \frac{1}{2} \times \sqrt{13} \times \sqrt{7} \times \frac{5\sqrt{3}}{\sqrt{91}} = \boldsymbol{\frac{5\sqrt{3}}{2}}$$

(2) 正四面体 O − ABC の体積は，

$$\frac{\sqrt{2}}{12} \times 6^3 = 18\sqrt{2}$$

$\dfrac{\mathrm{O-LMN}}{\mathrm{O-ABC}} = \dfrac{3 \cdot 4 \cdot 2}{6 \cdot 6 \cdot 6} = \dfrac{1}{9}$ だから，四面体 O − LMN の体積は，

$$\frac{1}{9} \times 18\sqrt{2} = \boldsymbol{2\sqrt{2}}$$

＜参考＞ (2) の $\dfrac{\mathrm{O-LMN}}{\mathrm{O-ABC}}$ について

(1) の図より，四面体 A − OBC と 四面体 A − OBN は底面の面積比より，

$$\frac{\mathrm{A-OBN}}{\mathrm{A-OBC}} = \frac{\triangle \mathrm{OBN}}{\triangle \mathrm{OBC}} = \frac{\mathrm{ON}}{\mathrm{OC}} = \frac{1}{3}$$

同様に，$\dfrac{\mathrm{N-OLB}}{\mathrm{N-OAB}} = \dfrac{1}{2}$，$\dfrac{\mathrm{L-OMN}}{\mathrm{L-OBN}} = \dfrac{2}{3}$ より，

$$\frac{\mathrm{O-LMN}}{\mathrm{O-ABC}} = \frac{\mathrm{A-OBN}}{\mathrm{A-OBC}} \cdot \frac{\mathrm{N-OLB}}{\mathrm{N-OAB}} \cdot \frac{\mathrm{L-OMN}}{\mathrm{L-OBN}} = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{9}$$

である．

(3)

$$\triangle \mathrm{OLM} = \frac{1}{2} \cdot 3 \cdot 4 \cdot \sin 60^\circ = 3\sqrt{3}$$
$$\triangle \mathrm{OMN} = \frac{1}{2} \cdot 4 \cdot 2 \cdot \sin 60^\circ = 2\sqrt{3}$$
$$\triangle \mathrm{ONL} = \frac{1}{2} \cdot 2 \cdot 3 \cdot \sin 60^\circ = \frac{3\sqrt{3}}{2}$$

だから，四面体 O − LMN の表面積は，

$$3\sqrt{3} + 2\sqrt{3} + \frac{3\sqrt{3}}{2} + \frac{5\sqrt{3}}{2} = 9\sqrt{3}$$

だから，四面体 O − LMN に内接する球の半径を r とすると，

$$\frac{1}{3} \cdot 9\sqrt{3} \cdot r = 2\sqrt{2} \qquad \therefore\ r = \boldsymbol{\frac{2\sqrt{6}}{9}}$$

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